NAG Library Routine Document

S21BFF

Note: before using this routine, please read the Users' Note for your implementation to check the interpretation of ***bold italicised*** terms and other implementation-dependent details.

1 Purpose

S21BFF returns a value of the classical (Legendre) form of the incomplete elliptic integral of the second kind, via the function name.

2 Specification

```
FUNCTION S21BFF (PHI, DM, IFAIL)
REAL (KIND=nag_wp) S21BFF

INTEGER            IFAIL
REAL (KIND=nag_wp) PHI, DM
```

3 Description

S21BFF calculates an approximation to the integral

$$E(\phi \mid m) = \int_0^{\phi} \left(1 - m\sin^2\theta\right)^{\frac{1}{2}} d\theta,$$

where $0 \leq \phi \leq \frac{\pi}{2}$ and $m\sin^2\phi \leq 1$.

The integral is computed using the symmetrised elliptic integrals of Carlson (Carlson (1979) and Carlson (1988)). The relevant identity is

$$E(\phi \mid m) = \sin\phi R_F(q,r,1) - \frac{1}{3}m\sin^3\phi R_D(q,r,1),$$

where $q = \cos^2\phi$, $r = 1 - m\sin^2\phi$, R_F is the Carlson symmetrised incomplete elliptic integral of the first kind (see S21BBF) and R_D is the Carlson symmetrised incomplete elliptic integral of the second kind (see S21BCF).

4 References

Abramowitz M and Stegun I A (1972) *Handbook of Mathematical Functions* (3rd Edition) Dover Publications

Carlson B C (1979) Computing elliptic integrals by duplication *Numerische Mathematik* **33** 1–16

Carlson B C (1988) A table of elliptic integrals of the third kind *Math. Comput.* **51** 267–280

5 Arguments

1: PHI – REAL (KIND=nag_wp) *Input*
2: DM – REAL (KIND=nag_wp) *Input*

On entry: the arguments ϕ and m of the function.

Constraints:

$0.0 \leq \text{PHI} \leq \frac{\pi}{2}$;
$\text{DM} \times \sin^2(\text{PHI}) \leq 1.0$.

3: IFAIL – INTEGER *Input/Output*

On entry: IFAIL must be set to 0, -1 or 1. If you are unfamiliar with this argument you should refer to Section 3.4 in How to Use the NAG Library and its Documentation for details.

For environments where it might be inappropriate to halt program execution when an error is detected, the value -1 or 1 is recommended. If the output of error messages is undesirable, then the value 1 is recommended. Otherwise, if you are not familiar with this argument, the recommended value is 0. **When the value -1 or 1 is used it is essential to test the value of IFAIL on exit.**

On exit: $\text{IFAIL} = 0$ unless the routine detects an error or a warning has been flagged (see Section 6).

6 Error Indicators and Warnings

If on entry $\text{IFAIL} = 0$ or -1, explanatory error messages are output on the current error message unit (as defined by X04AAF).

Errors or warnings detected by the routine:

$\text{IFAIL} = 1$

On entry, $\text{PHI} = \langle value \rangle$.
Constraint: $0 \leq \text{PHI} \leq \frac{\pi}{2}$.

$\text{IFAIL} = 2$

On entry, $\text{PHI} = \langle value \rangle$ and $\text{DM} = \langle value \rangle$; the integral is undefined.
Constraint: $\text{DM} \times \sin^2(\text{PHI}) \leq 1.0$.

$\text{IFAIL} = -99$

An unexpected error has been triggered by this routine. Please contact NAG.

See Section 3.9 in How to Use the NAG Library and its Documentation for further information.

$\text{IFAIL} = -399$

Your licence key may have expired or may not have been installed correctly.

See Section 3.8 in How to Use the NAG Library and its Documentation for further information.

$\text{IFAIL} = -999$

Dynamic memory allocation failed.

See Section 3.7 in How to Use the NAG Library and its Documentation for further information.

7 Accuracy

In principle S21BFF is capable of producing full ***machine precision***. However round-off errors in internal arithmetic will result in slight loss of accuracy. This loss should never be excessive as the algorithm does not involve any significant amplification of round-off error. It is reasonable to assume that the result is accurate to within a small multiple of the ***machine precision***.

8 Parallelism and Performance

S21BFF is not threaded in any implementation.

9 Further Comments

You should consult the S Chapter Introduction, which shows the relationship between this routine and the Carlson definitions of the elliptic integrals. In particular, the relationship between the argument-constraints for both forms becomes clear.

For more information on the algorithms used to compute R_F and R_D, see the routine documents for S21BBF and S21BCF, respectively.

If you wish to input a value of PHI outside the range allowed by this routine you should refer to Section 17.4 of Abramowitz and Stegun (1972) for useful identities. For example, $E(-\phi|m) = -E(\phi|m)$. A parameter $m > 1$ can be replaced by one less than unity using $E(\phi|m) = \sqrt{m}E\big(\phi\sqrt{m}|\frac{1}{m}\big) - (m-1)\phi$.

10 Example

This example simply generates a small set of nonextreme arguments that are used with the routine to produce the table of results.

10.1 Program Text

```
    Program s21bffe

!     S21BFF Example Program Text

!     Mark 26 Release. NAG Copyright 2016.

!     .. Use Statements ..
      Use nag_library, Only: nag_wp, s21bff, x01aaf
!     .. Implicit None Statement ..
      Implicit None
!     .. Parameters ..
      Integer, Parameter               :: nout = 6
!     .. Local Scalars ..
      Real (Kind=nag_wp)               :: dm, f, phi, pi
      Integer                          :: ifail, ix
!     .. Intrinsic Procedures ..
      Intrinsic                        :: real
!     .. Executable Statements ..
      Write (nout,*) 'S21BFF Example Program Results'

      Write (nout,*)
      Write (nout,*) '    PHI    DM      S21BFF'
      Write (nout,*)

      pi = x01aaf(pi)

data: Do ix = 1, 3
        phi = real(ix,kind=nag_wp)*pi/6.0E0_nag_wp
        dm = real(ix,kind=nag_wp)*0.25E0_nag_wp

        ifail = -1
        f = s21bff(phi,dm,ifail)

        If (ifail<0) Then
          Exit data
        End If

        Write (nout,99999) phi, dm, f
      End Do data

99999 Format (1X,2F7.2,F12.4)
    End Program s21bffe
```

10.2 Program Data

None.

10.3 Program Results

```
 S21BFF Example Program Results

     PHI     DM       S21BFF

    0.52   0.25       0.5179
    1.05   0.50       0.9650
    1.57   0.75       1.2111
```

Example Program
Classical (Legendre) Form of the Incomplete Elliptic Integral of the Second Kind