

# NAG Library Routine Document

## S17AHF

**Note:** before using this routine, please read the Users' Note for your implementation to check the interpretation of ***bold italicised*** terms and other implementation-dependent details.

### 1 Purpose

S17AHF returns a value of the Airy function,  $\text{Bi}(x)$ , via the function name.

### 2 Specification

```
FUNCTION S17AHF (X, IFAIL)
REAL (KIND=nag_wp) S17AHF
INTEGER IFAIL
REAL (KIND=nag_wp) X
```

### 3 Description

S17AHF evaluates an approximation to the Airy function  $\text{Bi}(x)$ . It is based on a number of Chebyshev expansions.

For  $x < -5$ ,

$$\text{Bi}(x) = \frac{a(t) \cos z + b(t) \sin z}{(-x)^{1/4}},$$

where  $z = \frac{\pi}{4} + \frac{2}{3}\sqrt{-x^3}$  and  $a(t)$  and  $b(t)$  are expansions in the variable  $t = -2\left(\frac{5}{x}\right)^3 - 1$ .

For  $-5 \leq x \leq 0$ ,

$$\text{Bi}(x) = \sqrt{3}(f(t) + xg(t)),$$

where  $f$  and  $g$  are expansions in  $t = -2\left(\frac{x}{5}\right)^3 - 1$ .

For  $0 < x < 4.5$ ,

$$\text{Bi}(x) = e^{11x/8}y(t),$$

where  $y$  is an expansion in  $t = 4x/9 - 1$ .

For  $4.5 \leq x < 9$ ,

$$\text{Bi}(x) = e^{5x/2}v(t),$$

where  $v$  is an expansion in  $t = 4x/9 - 3$ .

For  $x \geq 9$ ,

$$\text{Bi}(x) = \frac{e^z u(t)}{x^{1/4}},$$

where  $z = \frac{2}{3}\sqrt{x^3}$  and  $u$  is an expansion in  $t = 2\left(\frac{18}{x}\right) - 1$ .

For  $|x| < \textit{machine precision}$ , the result is set directly to  $\text{Bi}(0)$ . This both saves time and avoids possible intermediate underflows.

For large negative arguments, it becomes impossible to calculate the phase of the oscillating function with any accuracy so the routine must fail. This occurs if  $x < -\left(\frac{3}{2\epsilon}\right)^{2/3}$ , where  $\epsilon$  is the **machine precision**.

For large positive arguments, there is a danger of causing overflow since Bi grows in an essentially exponential manner, so the routine must fail.

## 4 References

Abramowitz M and Stegun I A (1972) *Handbook of Mathematical Functions* (3rd Edition) Dover Publications

## 5 Arguments

1: X – REAL (KIND=nag\_wp) *Input*

*On entry:* the argument  $x$  of the function.

2: IFAIL – INTEGER *Input/Output*

*On entry:* IFAIL must be set to 0, -1 or 1. If you are unfamiliar with this argument you should refer to Section 3.4 in How to Use the NAG Library and its Documentation for details.

For environments where it might be inappropriate to halt program execution when an error is detected, the value -1 or 1 is recommended. If the output of error messages is undesirable, then the value 1 is recommended. Otherwise, if you are not familiar with this argument, the recommended value is 0. **When the value -1 or 1 is used it is essential to test the value of IFAIL on exit.**

*On exit:* IFAIL = 0 unless the routine detects an error or a warning has been flagged (see Section 6).

## 6 Error Indicators and Warnings

If on entry IFAIL = 0 or -1, explanatory error messages are output on the current error message unit (as defined by X04AAF).

Errors or warnings detected by the routine:

IFAIL = 1

X is too large and positive. On softfailure, the routine returns zero. (see the Users' Note for your implementation for details)

IFAIL = 2

X is too large and negative. On softfailure, the routine returns zero. See also the Users' Note for your implementation.

IFAIL = -99

An unexpected error has been triggered by this routine. Please contact NAG.

See Section 3.9 in How to Use the NAG Library and its Documentation for further information.

IFAIL = -399

Your licence key may have expired or may not have been installed correctly.

See Section 3.8 in How to Use the NAG Library and its Documentation for further information.

IFAIL = -999

Dynamic memory allocation failed.

See Section 3.7 in How to Use the NAG Library and its Documentation for further information.

## 7 Accuracy

For negative arguments the function is oscillatory and hence absolute error is the appropriate measure. In the positive region the function is essentially exponential-like and here relative error is appropriate. The absolute error,  $E$ , and the relative error,  $\epsilon$ , are related in principle to the relative error in the argument,  $\delta$ , by

$$E \simeq |x\text{Bi}'(x)|\delta, \epsilon \simeq \left| \frac{x\text{Bi}'(x)}{\text{Bi}(x)} \right| \delta.$$

In practice, approximate equality is the best that can be expected. When  $\delta$ ,  $\epsilon$  or  $E$  is of the order of the *machine precision*, the errors in the result will be somewhat larger.

For small  $x$ , errors are strongly damped and hence will be bounded essentially by the *machine precision*.

For moderate to large negative  $x$ , the error behaviour is clearly oscillatory but the amplitude of the error grows like amplitude  $\left(\frac{E}{\delta}\right) \sim \frac{|x|^{5/4}}{\sqrt{\pi}}$ .

However the phase error will be growing roughly as  $\frac{2}{3}\sqrt{|x|^3}$  and hence all accuracy will be lost for large negative arguments. This is due to the impossibility of calculating sin and cos to any accuracy if  $\frac{2}{3}\sqrt{|x|^3} > \frac{1}{\delta}$ .

For large positive arguments, the relative error amplification is considerable:

$$\frac{\epsilon}{\delta} \sim \sqrt{x^3}.$$

This means a loss of roughly two decimal places accuracy for arguments in the region of 20. However very large arguments are not possible due to the danger of causing overflow and errors are therefore limited in practice.

## 8 Parallelism and Performance

S17AHF is not threaded in any implementation.

## 9 Further Comments

None.

## 10 Example

This example reads values of the argument  $x$  from a file, evaluates the function at each value of  $x$  and prints the results.

### 10.1 Program Text

```

Program s17ahfe

!      S17AHF Example Program Text

!      Mark 26 Release. NAG Copyright 2016.

!      .. Use Statements ..
```

```

      Use nag_library, Only: nag_wp, s17ahf
!      .. Implicit None Statement ..
      Implicit None
!      .. Parameters ..
      Integer, Parameter          :: nin = 5, nout = 6
!      .. Local Scalars ..
      Real (Kind=nag_wp)          :: x, y
      Integer                     :: ifail, ioerr
!      .. Executable Statements ..
      Write (nout,*) 'S17AHF Example Program Results'

!      Skip heading in data file
      Read (nin,*)

      Write (nout,*)
      Write (nout,*) '      X      Y'
      Write (nout,*)

data: Do
      Read (nin,*,Iostat=ioerr) x

      If (ioerr<0) Then
        Exit data
      End If

      ifail = -1
      y = s17ahf(x,ifail)

      If (ifail<0) Then
        Exit data
      End If

      Write (nout,99999) x, y
    End Do data

99999 Format (1X,1P,2E12.3)
End Program s17ahfe

```

## 10.2 Program Data

S17AHF Example Program Data

```

-10.0
-1.0
0.0
1.0
5.0
10.0
20.0

```

## 10.3 Program Results

S17AHF Example Program Results

| X          | Y          |
|------------|------------|
| -1.000E+01 | -3.147E-01 |
| -1.000E+00 | 1.040E-01  |
| 0.000E+00  | 6.149E-01  |
| 1.000E+00  | 1.207E+00  |
| 5.000E+00  | 6.578E+02  |
| 1.000E+01  | 4.556E+08  |
| 2.000E+01  | 2.104E+25  |

**Example Program**  
Returns a Value of the Airy Function  $Bi(x)$

