NAG Library Function Document

nag_elliptic_integral_E (s21bfc)

1 Purpose

nag_elliptic_integral_E (s21bfc) returns a value of the classical (Legendre) form of the incomplete elliptic integral of the second kind.

2 Specification

```
#include <nag.h>
#include <nags.h>
double nag_elliptic_integral_E (double phi, double dm, NagError *fail)
```

3 Description

nag_elliptic_integral_E (s21bfc) calculates an approximation to the integral

$$E(\phi \mid m) = \int_0^\phi \left(1 - m\sin^2\theta\right)^{\frac{1}{2}} d\theta,$$

where $0 \leq \phi \leq \frac{\pi}{2}$ and $m\sin^2\phi \leq 1$.

The integral is computed using the symmetrised elliptic integrals of Carlson (Carlson (1979) and Carlson (1988)). The relevant identity is

$$E(\phi \mid m) = \sin\phi R_F(q, r, 1) - \frac{1}{3} m \sin^3\phi R_D(q, r, 1),$$

where $q = \cos^2\phi$, $r = 1 - m\sin^2\phi$, R_F is the Carlson symmetrised incomplete elliptic integral of the first kind (see nag_elliptic_integral_rf (s21bbc)) and R_D is the Carlson symmetrised incomplete elliptic integral of the second kind (see nag_elliptic_integral_rd (s21bcc)).

4 References

Abramowitz M and Stegun I A (1972) *Handbook of Mathematical Functions* (3rd Edition) Dover Publications

Carlson B C (1979) Computing elliptic integrals by duplication *Numerische Mathematik* **33** 1–16

Carlson B C (1988) A table of elliptic integrals of the third kind *Math. Comput.* **51** 267–280

5 Arguments

1: **phi** – double *Input*

2: **dm** – double *Input*

On entry: the arguments ϕ and m of the function.

Constraints:

$0.0 \leq$ **phi** $\leq \frac{\pi}{2}$;
dm $\times \sin^2($**phi**$) \leq 1.0$.

3: **fail** – NagError * *Input/Output*

The NAG error argument (see Section 2.7 in How to Use the NAG Library and its Documentation).

6 Error Indicators and Warnings

NE_ALLOC_FAIL

Dynamic memory allocation failed.
See Section 2.3.1.2 in How to Use the NAG Library and its Documentation for further information.

NE_INTERNAL_ERROR

An internal error has occurred in this function. Check the function call and any array sizes. If the call is correct then please contact NAG for assistance.

An unexpected error has been triggered by this function. Please contact NAG.
See Section 2.7.6 in How to Use the NAG Library and its Documentation for further information.

NE_NO_LICENCE

Your licence key may have expired or may not have been installed correctly.
See Section 2.7.5 in How to Use the NAG Library and its Documentation for further information.

NE_REAL

On entry, **phi** $= \langle value \rangle$.
Constraint: $0 \leq$ **phi** $\leq \frac{\pi}{2}$.

NE_REAL_2

On entry, **phi** $= \langle value \rangle$ and **dm** $= \langle value \rangle$; the integral is undefined.
Constraint: **dm** $\times \sin^2($**phi**$) \leq 1.0$.

7 Accuracy

In principle nag_elliptic_integral_E (s21bfc) is capable of producing full ***machine precision***. However round-off errors in internal arithmetic will result in slight loss of accuracy. This loss should never be excessive as the algorithm does not involve any significant amplification of round-off error. It is reasonable to assume that the result is accurate to within a small multiple of the ***machine precision***.

8 Parallelism and Performance

nag_elliptic_integral_E (s21bfc) is not threaded in any implementation.

9 Further Comments

You should consult the s Chapter Introduction, which shows the relationship between this function and the Carlson definitions of the elliptic integrals. In particular, the relationship between the argument-constraints for both forms becomes clear.

For more information on the algorithms used to compute R_F and R_D, see the function documents for nag_elliptic_integral_rf (s21bbc) and nag_elliptic_integral_rd (s21bcc), respectively.

If you wish to input a value of **phi** outside the range allowed by this function you should refer to Section 17.4 of Abramowitz and Stegun (1972) for useful identities. For example, $E(-\phi|m) = -E(\phi|m)$. A parameter $m > 1$ can be replaced by one less than unity using $E(\phi|m) = \sqrt{m}E\left(\phi\sqrt{m}|\frac{1}{m}\right) - (m-1)\phi$.

10 Example

This example simply generates a small set of nonextreme arguments that are used with the function to produce the table of results.

10.1 Program Text

```
/* nag_elliptic_integral_E (s21bfc) Example Program.
 *
 * NAGPRODCODE Version.
 *
 * Copyright 2016 Numerical Algorithms Group.
 *
 * Mark 26, 2016.
 */
/* Pre-processor includes */
#include <stdio.h>
#include <nag.h>
#include <nag_stdlib.h>
#include <nags.h>
#include <nagx01.h>

int main(void)
{
  /*Integer scalar and array declarations */
  Integer exit_status = 0;
  Integer ix;
  /*Double scalar and array declarations */
  double dm, e, phi, pi;
  NagError fail;

  INIT_FAIL(fail);

  printf("nag_elliptic_integral_E (s21bfc) Example Program Results\n");
  printf("\n    phi    dm       nag_elliptic_integral_E\n\n");
  pi = nag_pi;
  for (ix = 1; ix <= 3; ix++) {
    phi = ix * pi / 6.00e0;
    dm = ix * 0.250e0;
    /*
     * nag_elliptic_integral_E (s21bfc)
     * Elliptic integral of 2nd kind, Legendre form, E( phi |m)
     */
    e = nag_elliptic_integral_E(phi, dm, &fail);
    if (fail.code != NE_NOERROR) {
      printf("Error from nag_elliptic_integral_E (s21bfc).\n%s\n",
             fail.message);
      exit_status = 1;
      goto END;
    }
    printf("%7.2f%7.2f%12.4f\n", phi, dm, e);
  }

END:

  return exit_status;
}
```

10.2 Program Data

None.

10.3 Program Results

```
nag_elliptic_integral_E (s21bfc) Example Program Results

    phi    dm       nag_elliptic_integral_E

   0.52   0.25      0.5179
   1.05   0.50      0.9650
   1.57   0.75      1.2111
```

Example Program
Classical (Legendre) Form of the Incomplete Elliptic Integral of the Second Kind