

# NAG Library Function Document

## nag\_dggrqf (f08zfc)

### 1 Purpose

nag\_dggrqf (f08zfc) computes a generalized  $RQ$  factorization of a real matrix pair  $(A, B)$ , where  $A$  is an  $m$  by  $n$  matrix and  $B$  is a  $p$  by  $n$  matrix.

### 2 Specification

```
#include <nag.h>
#include <nagf08.h>

void nag_dggrqf (Nag_OrderType order, Integer m, Integer p, Integer n,
                 double a[], Integer pda, double taua[], double b[], Integer pdb,
                 double taub[], NagError *fail)
```

### 3 Description

nag\_dggrqf (f08zfc) forms the generalized  $RQ$  factorization of an  $m$  by  $n$  matrix  $A$  and a  $p$  by  $n$  matrix  $B$

$$A = RQ, \quad B = ZTQ,$$

where  $Q$  is an  $n$  by  $n$  orthogonal matrix,  $Z$  is a  $p$  by  $p$  orthogonal matrix and  $R$  and  $T$  are of the form

$$R = \begin{cases} m \begin{pmatrix} n-m & m \\ & 0 & R_{12} \end{pmatrix}; & \text{if } m \leq n, \\ m-n \begin{pmatrix} n \\ R_{11} \\ n \\ R_{21} \end{pmatrix}; & \text{if } m > n, \end{cases}$$

with  $R_{12}$  or  $R_{21}$  upper triangular,

$$T = \begin{cases} p-n \begin{pmatrix} n \\ T_{11} \\ 0 \end{pmatrix}; & \text{if } p \geq n, \\ p \begin{pmatrix} n-p & \\ T_{11} & T_{12} \end{pmatrix}; & \text{if } p < n, \end{cases}$$

with  $T_{11}$  upper triangular.

In particular, if  $B$  is square and nonsingular, the generalized  $RQ$  factorization of  $A$  and  $B$  implicitly gives the  $RQ$  factorization of  $AB^{-1}$  as

$$AB^{-1} = (RT^{-1})Z^T.$$

### 4 References

Anderson E, Bai Z, Bischof C, Blackford S, Demmel J, Dongarra J J, Du Croz J J, Greenbaum A, Hammarling S, McKenney A and Sorensen D (1999) *LAPACK Users' Guide* (3rd Edition) SIAM, Philadelphia <http://www.netlib.org/lapack/lug>

Anderson E, Bai Z and Dongarra J (1992) Generalized  $QR$  factorization and its applications *Linear Algebra Appl. (Volume 162–164)* 243–271

Hammarling S (1987) The numerical solution of the general Gauss-Markov linear model *Mathematics in Signal Processing* (eds T S Durrani, J B Abbiss, J E Hudson, R N Madan, J G McWhirter and T A Moore) 441–456 Oxford University Press

Paige C C (1990) Some aspects of generalized *QR* factorizations . *In Reliable Numerical Computation* (eds M G Cox and S Hammarling) 73–91 Oxford University Press

## 5 Arguments

1: **order** – Nag\_OrderType *Input*

*On entry:* the **order** argument specifies the two-dimensional storage scheme being used, i.e., row-major ordering or column-major ordering. C language defined storage is specified by **order** = Nag\_RowMajor. See Section 2.3.1.3 in How to Use the NAG Library and its Documentation for a more detailed explanation of the use of this argument.

*Constraint:* **order** = Nag\_RowMajor or Nag\_ColMajor.

2: **m** – Integer *Input*

*On entry:*  $m$ , the number of rows of the matrix  $A$ .

*Constraint:*  $m \geq 0$ .

3: **p** – Integer *Input*

*On entry:*  $p$ , the number of rows of the matrix  $B$ .

*Constraint:*  $p \geq 0$ .

4: **n** – Integer *Input*

*On entry:*  $n$ , the number of columns of the matrices  $A$  and  $B$ .

*Constraint:*  $n \geq 0$ .

5: **a**[*dim*] – double *Input/Output*

**Note:** the dimension, *dim*, of the array **a** must be at least

$\max(1, \mathbf{pda} \times \mathbf{n})$  when **order** = Nag\_ColMajor;  
 $\max(1, \mathbf{m} \times \mathbf{pda})$  when **order** = Nag\_RowMajor.

Where  $\mathbf{A}(i, j)$  appears in this document, it refers to the array element

$\mathbf{a}[(j-1) \times \mathbf{pda} + i - 1]$  when **order** = Nag\_ColMajor;  
 $\mathbf{a}[(i-1) \times \mathbf{pda} + j - 1]$  when **order** = Nag\_RowMajor.

*On entry:* the  $m$  by  $n$  matrix  $A$ .

*On exit:* if  $m \leq n$ , the upper triangle of the subarray  $\mathbf{A}(1 : m, n - m + 1 : n)$  contains the  $m$  by  $m$  upper triangular matrix  $R_{12}$ .

If  $m \geq n$ , the elements on and above the  $(m - n)$ th subdiagonal contain the  $m$  by  $n$  upper trapezoidal matrix  $R$ ; the remaining elements, with the array **taua**, represent the orthogonal matrix  $Q$  as a product of  $\min(m, n)$  elementary reflectors (see Section 3.3.6 in the f08 Chapter Introduction).

6: **pda** – Integer *Input*

*On entry:* the stride separating row or column elements (depending on the value of **order**) in the array **a**.

*Constraints:*

if **order** = Nag\_ColMajor,  $\mathbf{pda} \geq \max(1, \mathbf{m})$ ;  
 if **order** = Nag\_RowMajor,  $\mathbf{pda} \geq \max(1, \mathbf{n})$ .

- 7: **taua**[**min**(**m**, **n**)] – double *Output*  
*On exit:* the scalar factors of the elementary reflectors which represent the orthogonal matrix  $Q$ .
- 8: **b**[*dim*] – double *Input/Output*  
**Note:** the dimension, *dim*, of the array **b** must be at least  
 $\max(1, \mathbf{pdb} \times \mathbf{n})$  when **order** = Nag\_ColMajor;  
 $\max(1, \mathbf{p} \times \mathbf{pdb})$  when **order** = Nag\_RowMajor.  
The (*i*, *j*)th element of the matrix  $B$  is stored in  
 $\mathbf{b}[(j-1) \times \mathbf{pdb} + i - 1]$  when **order** = Nag\_ColMajor;  
 $\mathbf{b}[(i-1) \times \mathbf{pdb} + j - 1]$  when **order** = Nag\_RowMajor.  
*On entry:* the  $p$  by  $n$  matrix  $B$ .  
*On exit:* the elements on and above the diagonal of the array contain the  $\min(p, n)$  by  $n$  upper trapezoidal matrix  $T$  ( $T$  is upper triangular if  $p \geq n$ ); the elements below the diagonal, with the array **taub**, represent the orthogonal matrix  $Z$  as a product of elementary reflectors (see Section 3.3.6 in the f08 Chapter Introduction).
- 9: **pdb** – Integer *Input*  
*On entry:* the stride separating row or column elements (depending on the value of **order**) in the array **b**.  
*Constraints:*  
if **order** = Nag\_ColMajor, **pdb**  $\geq \max(1, \mathbf{p})$ ;  
if **order** = Nag\_RowMajor, **pdb**  $\geq \max(1, \mathbf{n})$ .
- 10: **taub**[**min**(**p**, **n**)] – double *Output*  
*On exit:* the scalar factors of the elementary reflectors which represent the orthogonal matrix  $Z$ .
- 11: **fail** – NagError \* *Input/Output*  
The NAG error argument (see Section 2.7 in How to Use the NAG Library and its Documentation).

## 6 Error Indicators and Warnings

### NE\_ALLOC\_FAIL

Dynamic memory allocation failed.

See Section 2.3.1.2 in How to Use the NAG Library and its Documentation for further information.

### NE\_BAD\_PARAM

On entry, argument  $\langle value \rangle$  had an illegal value.

### NE\_INT

On entry, **m** =  $\langle value \rangle$ .

Constraint: **m**  $\geq 0$ .

On entry, **n** =  $\langle value \rangle$ .

Constraint: **n**  $\geq 0$ .

On entry, **p** =  $\langle value \rangle$ .

Constraint: **p**  $\geq 0$ .

On entry, **pda** =  $\langle value \rangle$ .

Constraint: **pda**  $> 0$ .

On entry, **pdb** =  $\langle value \rangle$ .  
 Constraint: **pdb** > 0.

## NE\_INT\_2

On entry, **pda** =  $\langle value \rangle$  and **m** =  $\langle value \rangle$ .  
 Constraint: **pda**  $\geq$  max(1, **m**).

On entry, **pda** =  $\langle value \rangle$  and **n** =  $\langle value \rangle$ .  
 Constraint: **pda**  $\geq$  max(1, **n**).

On entry, **pdb** =  $\langle value \rangle$  and **n** =  $\langle value \rangle$ .  
 Constraint: **pdb**  $\geq$  max(1, **n**).

On entry, **pdb** =  $\langle value \rangle$  and **p** =  $\langle value \rangle$ .  
 Constraint: **pdb**  $\geq$  max(1, **p**).

## NE\_INTERNAL\_ERROR

An internal error has occurred in this function. Check the function call and any array sizes. If the call is correct then please contact NAG for assistance.

An unexpected error has been triggered by this function. Please contact NAG.  
 See Section 2.7.6 in How to Use the NAG Library and its Documentation for further information.

## NE\_NO\_LICENCE

Your licence key may have expired or may not have been installed correctly.  
 See Section 2.7.5 in How to Use the NAG Library and its Documentation for further information.

## 7 Accuracy

The computed generalized  $RQ$  factorization is the exact factorization for nearby matrices  $(A + E)$  and  $(B + F)$ , where

$$\|E\|_2 = O\epsilon\|A\|_2 \quad \text{and} \quad \|F\|_2 = O\epsilon\|B\|_2,$$

and  $\epsilon$  is the *machine precision*.

## 8 Parallelism and Performance

nag\_dggrqf (f08zfc) is threaded by NAG for parallel execution in multithreaded implementations of the NAG Library.

nag\_dggrqf (f08zfc) makes calls to BLAS and/or LAPACK routines, which may be threaded within the vendor library used by this implementation. Consult the documentation for the vendor library for further information.

Please consult the x06 Chapter Introduction for information on how to control and interrogate the OpenMP environment used within this function. Please also consult the Users' Note for your implementation for any additional implementation-specific information.

## 9 Further Comments

The orthogonal matrices  $Q$  and  $Z$  may be formed explicitly by calls to nag\_dorgqr (f08cjc) and nag\_dorgqr (f08afc) respectively. nag\_dormqr (f08ckc) may be used to multiply  $Q$  by another matrix and nag\_dormqr (f08agc) may be used to multiply  $Z$  by another matrix.

The complex analogue of this function is nag\_zggrqf (f08ztc).

## 10 Example

This example solves the least squares problem

$$\underset{x}{\text{minimize}} \|c - Ax\|_2 \quad \text{subject to} \quad Bx = d$$

where

$$A = \begin{pmatrix} -0.57 & -1.28 & -0.39 & 0.25 \\ -1.93 & 1.08 & -0.31 & -2.14 \\ 2.30 & 0.24 & 0.40 & -0.35 \\ -1.93 & 0.64 & -0.66 & 0.08 \\ 0.15 & 0.30 & 0.15 & -2.13 \\ -0.02 & 1.03 & -1.43 & 0.50 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix},$$

$$c = \begin{pmatrix} -1.50 \\ -2.14 \\ 1.23 \\ -0.54 \\ -1.68 \\ 0.82 \end{pmatrix} \quad \text{and} \quad d = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The constraints  $Bx = d$  correspond to  $x_1 = x_3$  and  $x_2 = x_4$ .

The solution is obtained by first computing a generalized  $RQ$  factorization of the matrix pair  $(B, A)$ . The example illustrates the general solution process.

### 10.1 Program Text

```
/* nag_dggrqf (f08zfc) Example Program.
*
* NAGPRODCODE Version.
*
* Copyright 2016 Numerical Algorithms Group.
*
* Mark 26, 2016.
*/

#include <stdio.h>
#include <nag.h>
#include <nag_stdlib.h>
#include <nagf07.h>
#include <nagf08.h>
#include <nagf16.h>

int main(void)
{
    /* Scalars */
    double alpha, beta, rnorm;
    Integer i, j, m, n, p, pda, pdb, pdc, pdd, pdx;
    Integer ylrows, y2rows, y3rows;
    Integer exit_status = 0;

    /* Arrays */
    double *a = 0, *b = 0, *c = 0, *d = 0, *taua = 0, *taub = 0, *x = 0;

    /* Nag Types */
    NagError fail;
    Nag_OrderType order;

#ifdef NAG_COLUMN_MAJOR
#define A(I, J) a[(J-1)*pda + I - 1]
#define B(I, J) b[(J-1)*pdb + I - 1]
    order = Nag_ColMajor;
#else
#define A(I, J) a[(I-1)*pda + J - 1]
#define B(I, J) b[(I-1)*pdb + J - 1]
    order = Nag_RowMajor;
#endif
```

```

#endif

    INIT_FAIL(fail);

    printf("nag_dggrqf (f08zfc) Example Program Results\n\n");

    /* Skip heading in data file */
#ifdef _WIN32
    scanf_s("%*[\n]");
#else
    scanf("%*[\n]");
#endif
#ifdef _WIN32
    scanf_s("%" NAG_IFMT "%" NAG_IFMT "%" NAG_IFMT "%*[\n]", &m, &n, &p);
#else
    scanf("%" NAG_IFMT "%" NAG_IFMT "%" NAG_IFMT "%*[\n]", &m, &n, &p);
#endif
    if (n < 0 || m < 0 || p < 0) {
        printf("Invalid n, m or p\n");
        exit_status = 1;
        goto END;
    }

#ifdef NAG_COLUMN_MAJOR
    pda = m;
    pdb = p;
    pdc = m;
    pdd = p;
    pdx = n;
#else
    pda = n;
    pdb = n;
    pdc = 1;
    pdd = 1;
    pdx = 1;
#endif

    /* Allocate memory */
    if (!(a = NAG_ALLOC(m * n, double)) ||
        !(b = NAG_ALLOC(p * n, double)) ||
        !(c = NAG_ALLOC(m, double)) ||
        !(d = NAG_ALLOC(p, double)) ||
        !(taua = NAG_ALLOC(MIN(m, n), double)) ||
        !(taub = NAG_ALLOC(MIN(n, p), double)) || !(x = NAG_ALLOC(n, double)))
    {
        printf("Allocation failure\n");
        exit_status = -1;
        goto END;
    }

    /* Read A, B, c and d from data file for the problem
     * min{||c-Ax||_2, x in R^n and B x = d}.
     */
    for (i = 1; i <= m; ++i)
#ifdef _WIN32
        for (j = 1; j <= n; ++j)
            scanf_s("%lf", &A(i, j));
#else
        for (j = 1; j <= n; ++j)
            scanf("%lf", &A(i, j));
#endif
#ifdef _WIN32
    scanf_s("%*[\n]");
#else
    scanf("%*[\n]");
#endif
    for (i = 1; i <= p; ++i)
#ifdef _WIN32
        for (j = 1; j <= n; ++j)
            scanf_s("%lf", &B(i, j));
#else

```

```

        for (j = 1; j <= n; ++j)
            scanf("%lf", &B(i, j));
    #endif
    #ifdef _WIN32
        scanf_s("%*[^\\n]");
    #else
        scanf("%*[^\\n]");
    #endif
    #ifdef _WIN32
        for (i = 0; i < m; ++i)
            scanf_s("%lf", &c[i]);
    #else
        for (i = 0; i < m; ++i)
            scanf("%lf", &c[i]);
    #endif
    #ifdef _WIN32
        scanf_s("%*[^\\n]");
    #else
        scanf("%*[^\\n]");
    #endif
    #ifdef _WIN32
        for (i = 0; i < p; ++i)
            scanf_s("%lf", &d[i]);
    #else
        for (i = 0; i < p; ++i)
            scanf("%lf", &d[i]);
    #endif
    #ifdef _WIN32
        scanf_s("%*[^\\n]");
    #else
        scanf("%*[^\\n]");
    #endif

    /* First compute the generalized RQ factorization of (B,A) as
     * B = (0 R12)*Q,   A = Z*(T11 T12 T13)*Q = T*Q.
     *               ( 0  T22 T23)
     * where R12, T11 and T22 are upper triangular,
     * using nag_dggrqf (f08zfc).
     */
    nag_dggrqf(order, p, m, n, b, pdb, taub, a, pda, taua, &fail);
    if (fail.code != NE_NOERROR) {
        printf("Error from nag_dggrqf (f08zfc).\\n%s\\n", fail.message);
        exit_status = 1;
        goto END;
    }

    /* Now, trans(z)*(c-Ax) = trans(z)*c - T*Q*x, and
     * let f = (f1) = trans(z) * (c1) => minimize ||f - T*Q*x||
     *          (f2)          (c2)
     * Compute f using nag_dormqr (f08agc), storing result in c
     */
    nag_dormqr(order, Nag_LeftSide, Nag_Trans, m, 1, MIN(m, n), a, pda, taua,
               c, pdc, &fail);
    if (fail.code != NE_NOERROR) {
        printf("Error from nag_dormqr (f08agc).\\n%s\\n", fail.message);
        exit_status = 1;
        goto END;
    }

    /* Putting Q*x = (y1), B * x = d becomes (0 R12) (y1) = d;
     *               (w )                (w )
     * => R12 * w = d.
     * Solve for w using nag_dtrtrs (f07tec), storing result in d;
     * R12 is (p by p) triangular submatrix starting at B(1,n-p+1).
     */
    nag_dtrtrs(order, Nag_Upper, Nag_NoTrans, Nag_NonUnitDiag, p, 1,
               &B(1, n - p + 1), pdb, d, pdd, &fail);
    if (fail.code != NE_NOERROR) {
        printf("Error from nag_dtrtrs (f07tec).\\n%s\\n", fail.message);
        exit_status = 1;
        goto END;
    }

```

```

}

/* The problem now reduces to finding the minimum norm of
 *  $g = (g_1) = (f_1) - T_{11}y_1 - (T_{12} \ T_{13})w$ 
 *  $(g_2) \quad (f_2) \quad - (T_{22} \ T_{23})w.$ 
 * Form  $c_1 = f_1 - (T_{12} \ T_{13})w$  using nag_dgemv (f16pac).
 */
alpha = -1.0;
beta = 1.0;
ylrows = n - p;
nag_dgemv(order, Nag_NoTrans, ylrows, p, alpha, &A(1, n - p + 1), pda, d, 1,
          beta, c, 1, &fail);
if (fail.code != NE_NOERROR) {
    printf("Error from nag_dgemv (f16pac).\n%s\n", fail.message);
    exit_status = 1;
    goto END;
}

/* => now  $(g_1) = c - T_{11}y_1$  and  $||g_1|| = 0$  when  $T_{11} * y_1 = c_1$ .
 * Solve this for  $y_1$  using nag_dtrtrs (f07tec) storing result in  $c_1$ .
 */
nag_dtrtrs(order, Nag_Upper, Nag_NoTrans, Nag_NonUnitDiag, ylrows, 1, a,
           pda, c, pdc, &fail);
if (fail.code != NE_NOERROR) {
    printf("Error from nag_dtrtrs (f07tec).\n%s\n", fail.message);
    exit_status = 1;
    goto END;
}

/* So now  $Qx = (y_1)$  is stored in  $(c_1)$ , which is now copied to  $x$ .
 *  $(w) \quad (d)$ 
 */
for (i = 0; i < ylrows; ++i)
    x[i] = c[i];
for (i = ylrows; i < n; ++i)
    x[i] = d[i - ylrows];

/* Compute  $x$  by applying transpose of  $Q$  using nag_dormrq (f08ckc). */
nag_dormrq(order, Nag_LeftSide, Nag_Trans, n, 1, p, b, pdb, taub, x, pdx,
           &fail);
if (fail.code != NE_NOERROR) {
    printf("Error from nag_dormrq (f08ckc).\n%s\n", fail.message);
    exit_status = 1;
    goto END;
}

/* It remains to minimize  $||g_2||$ ,  $g_2 = f_2 - (T_{22} \ T_{23})w$ .
 * Putting  $w = (y_2)$ , gives  $g_2 = f_2 - T_{22}y_2 - T_{23}y_3$ 
 *  $(y_3)$ 
 * [ $y_2$  stored in  $d_1$ , first  $y_2$  rows of  $d$ ;  $y_3$  stored in  $d_2$ , next  $n-m$  rows of  $d$ .]
 *
 * First form  $T_{22}y_2$  using nag_dtrmv (f16pfc) where  $y_2$  is held in  $d$ .
 */
y2rows = MIN(m, n) - ylrows;
alpha = 1.0;
nag_dtrmv(order, Nag_Upper, Nag_NoTrans, Nag_NonUnitDiag, y2rows, alpha,
          &A(n - p + 1, n - p + 1), pda, d, 1, &fail);
if (fail.code != NE_NOERROR) {
    printf("Error from nag_dtrmv (f16pfc).\n%s\n", fail.message);
    exit_status = 1;
    goto END;
}

/* Then,  $f_2 - T_{22}y_2$  ( $c_2 = c_2 - d$ ) */
for (i = 0; i < y2rows; ++i)
    c[ylrows + i] -= d[i];
y2rows = m - ylrows;
if (m < n) {
    y3rows = n - m;
    /* Then  $g_2 = f_2 - T_{22}y_2 - T_{23}y_3$  ( $c_2 = c_2 - T_{23}d_2$ ) */
    alpha = -1.0;
    beta = 1.0;

```



```

    nag_dgemv(order, Nag_NoTrans, y2rows, y3rows, alpha, &A(n - p + 1, m + 1),
              pda, &d[y2rows], 1, beta, &c[y1rows], 1, &fail);
    if (fail.code != NE_NOERROR) {
        printf("Error from nag_dgemv (f16pac).\n%s\n", fail.message);
        exit_status = 1;
        goto END;
    }
}

/* Compute ||g|| = ||g2|| = norm(f2 - T22*y2 - T23*y3)
 * using nag_dge_norm (f16rac).
 */
nag_dge_norm(Nag_ColMajor, Nag_FrobeniusNorm, y2rows, 1, &c[y1rows], y2rows,
             &rnorm, &fail);
if (fail.code != NE_NOERROR) {
    printf("Error from nag_dge_norm (f16rac).\n%s\n", fail.message);
    exit_status = 1;
    goto END;
}

/* Print least squares solution x */
printf("Constrained least squares solution\n");
for (i = 0; i < n; ++i)
    printf(" %10.4f%s", x[i], i % 7 == 6 ? "\n" : "");

/* Print the square root of the residual sum of squares */
printf("\nSquare root of the residual sum of squares\n");
printf("%11.2e\n", rnorm);

END:
    NAG_FREE(a);
    NAG_FREE(b);
    NAG_FREE(c);
    NAG_FREE(d);
    NAG_FREE(atau);
    NAG_FREE(taub);
    NAG_FREE(x);

    return exit_status;
}

```

## 10.2 Program Data

nag\_dggrqf (f08zfc) Example Program Data

```

    6      4      2      : m, n and p

-0.57 -1.28 -0.39  0.25
-1.93  1.08 -0.31 -2.14
 2.30  0.24  0.40 -0.35
-1.93  0.64 -0.66  0.08
 0.15  0.30  0.15 -2.13
-0.02  1.03 -1.43  0.50 : matrix A[m*n]

 1.00  0.00 -1.00  0.00
 0.00  1.00  0.00 -1.00 : matrix B[p*n]

-1.50
-2.14
 1.23
-0.54
-1.68
 0.82      : vector c[m]

 0.00
 0.00      : vector d[p]

```

### 10.3 Program Results

nag\_dggrqf (f08zfc) Example Program Results

Constrained least squares solution

0.4890	0.9975	0.4890	0.9975
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Square root of the residual sum of squares

2.51e-02

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