

# NAG Library Function Document

## nag\_dspgvd (f08tcc)

### 1 Purpose

nag\_dspgvd (f08tcc) computes all the eigenvalues and, optionally, the eigenvectors of a real generalized symmetric-definite eigenproblem, of the form

$$Az = \lambda Bz, \quad ABz = \lambda z \quad \text{or} \quad BAz = \lambda z,$$

where  $A$  and  $B$  are symmetric, stored in packed format, and  $B$  is also positive definite. If eigenvectors are desired, it uses a divide-and-conquer algorithm.

### 2 Specification

```
#include <nag.h>
#include <nagf08.h>

void nag_dspgvd (Nag_OrderType order, Integer itype, Nag_JobType job,
                 Nag_UploType uplo, Integer n, double ap[], double bp[], double w[],
                 double z[], Integer pdz, NagError *fail)
```

### 3 Description

nag\_dspgvd (f08tcc) first performs a Cholesky factorization of the matrix  $B$  as  $B = U^T U$ , when **uplo** = Nag\_Upper or  $B = LL^T$ , when **uplo** = Nag\_Lower. The generalized problem is then reduced to a standard symmetric eigenvalue problem

$$Cx = \lambda x,$$

which is solved for the eigenvalues and, optionally, the eigenvectors; the eigenvectors are then backtransformed to give the eigenvectors of the original problem.

For the problem  $Az = \lambda Bz$ , the eigenvectors are normalized so that the matrix of eigenvectors,  $z$ , satisfies

$$Z^T A Z = \Lambda \quad \text{and} \quad Z^T B Z = I,$$

where  $\Lambda$  is the diagonal matrix whose diagonal elements are the eigenvalues. For the problem  $ABz = \lambda z$  we correspondingly have

$$Z^{-1} A Z^{-T} = \Lambda \quad \text{and} \quad Z^T B Z = I,$$

and for  $BAz = \lambda z$  we have

$$Z^T A Z = \Lambda \quad \text{and} \quad Z^T B^{-1} Z = I.$$

### 4 References

Anderson E, Bai Z, Bischof C, Blackford S, Demmel J, Dongarra J J, Du Croz J J, Greenbaum A, Hammarling S, McKenney A and Sorensen D (1999) *LAPACK Users' Guide* (3rd Edition) SIAM, Philadelphia <http://www.netlib.org/lapack/lug>

Golub G H and Van Loan C F (1996) *Matrix Computations* (3rd Edition) Johns Hopkins University Press, Baltimore

## 5 Arguments

- 1: **order** – Nag\_OrderType *Input*  
*On entry:* the **order** argument specifies the two-dimensional storage scheme being used, i.e., row-major ordering or column-major ordering. C language defined storage is specified by **order** = Nag\_RowMajor. See Section 2.3.1.3 in How to Use the NAG Library and its Documentation for a more detailed explanation of the use of this argument.  
*Constraint:* **order** = Nag\_RowMajor or Nag\_ColMajor.
- 2: **itype** – Integer *Input*  
*On entry:* specifies the problem type to be solved.  
**itype** = 1  
 $Az = \lambda Bz$ .  
**itype** = 2  
 $ABz = \lambda z$ .  
**itype** = 3  
 $BAz = \lambda z$ .  
*Constraint:* **itype** = 1, 2 or 3.
- 3: **job** – Nag\_JobType *Input*  
*On entry:* indicates whether eigenvectors are computed.  
**job** = Nag\_EigVals  
Only eigenvalues are computed.  
**job** = Nag\_DoBoth  
Eigenvalues and eigenvectors are computed.  
*Constraint:* **job** = Nag\_EigVals or Nag\_DoBoth.
- 4: **uplo** – Nag\_UploType *Input*  
*On entry:* if **uplo** = Nag\_Upper, the upper triangles of  $A$  and  $B$  are stored.  
If **uplo** = Nag\_Lower, the lower triangles of  $A$  and  $B$  are stored.  
*Constraint:* **uplo** = Nag\_Upper or Nag\_Lower.
- 5: **n** – Integer *Input*  
*On entry:*  $n$ , the order of the matrices  $A$  and  $B$ .  
*Constraint:* **n**  $\geq$  0.
- 6: **ap**[*dim*] – double *Input/Output*  
**Note:** the dimension, *dim*, of the array **ap** must be at least  $\max(1, \mathbf{n} \times (\mathbf{n} + 1)/2)$ .  
*On entry:* the upper or lower triangle of the  $n$  by  $n$  symmetric matrix  $A$ , packed by rows or columns.  
The storage of elements  $A_{ij}$  depends on the **order** and **uplo** arguments as follows:
  - if **order** = Nag\_ColMajor and **uplo** = Nag\_Upper,  
 $A_{ij}$  is stored in **ap**[( $j - 1$ )  $\times$   $j/2 + i - 1$ ], for  $i \leq j$ ;
  - if **order** = Nag\_ColMajor and **uplo** = Nag\_Lower,  
 $A_{ij}$  is stored in **ap**[( $2n - j$ )  $\times$  ( $j - 1$ )/2 +  $i - 1$ ], for  $i \geq j$ ;
  - if **order** = Nag\_RowMajor and **uplo** = Nag\_Upper,  
 $A_{ij}$  is stored in **ap**[( $2n - i$ )  $\times$  ( $i - 1$ )/2 +  $j - 1$ ], for  $i \leq j$ ;

if **order** = Nag\_RowMajor and **uplo** = Nag\_Lower,  
 $A_{ij}$  is stored in **ap** $[(i-1) \times i/2 + j-1]$ , for  $i \geq j$ .

On exit: the contents of **ap** are destroyed.

7: **bp** $[dim]$  – double Input/Output

**Note:** the dimension,  $dim$ , of the array **bp** must be at least  $\max(1, n \times (n+1)/2)$ .

On entry: the upper or lower triangle of the  $n$  by  $n$  symmetric matrix  $B$ , packed by rows or columns.

The storage of elements  $B_{ij}$  depends on the **order** and **uplo** arguments as follows:

if **order** = Nag\_ColMajor and **uplo** = Nag\_Upper,  
 $B_{ij}$  is stored in **bp** $[(j-1) \times j/2 + i-1]$ , for  $i \leq j$ ;  
 if **order** = Nag\_ColMajor and **uplo** = Nag\_Lower,  
 $B_{ij}$  is stored in **bp** $[(2n-j) \times (j-1)/2 + i-1]$ , for  $i \geq j$ ;  
 if **order** = Nag\_RowMajor and **uplo** = Nag\_Upper,  
 $B_{ij}$  is stored in **bp** $[(2n-i) \times (i-1)/2 + j-1]$ , for  $i \leq j$ ;  
 if **order** = Nag\_RowMajor and **uplo** = Nag\_Lower,  
 $B_{ij}$  is stored in **bp** $[(i-1) \times i/2 + j-1]$ , for  $i \geq j$ .

On exit: the triangular factor  $U$  or  $L$  from the Cholesky factorization  $B = U^T U$  or  $B = LL^T$ , in the same storage format as  $B$ .

8: **w** $[n]$  – double Output

On exit: the eigenvalues in ascending order.

9: **z** $[dim]$  – double Output

**Note:** the dimension,  $dim$ , of the array **z** must be at least

$\max(1, \mathbf{pdz} \times n)$  when **job** = Nag\_DoBoth;  
 1 otherwise.

The  $(i, j)$ th element of the matrix  $Z$  is stored in

**z** $[(j-1) \times \mathbf{pdz} + i-1]$  when **order** = Nag\_ColMajor;  
**z** $[(i-1) \times \mathbf{pdz} + j-1]$  when **order** = Nag\_RowMajor.

On exit: if **job** = Nag\_DoBoth, **z** contains the matrix  $Z$  of eigenvectors. The eigenvectors are normalized as follows:

if **itype** = 1 or 2,  $Z^T B Z = I$ ;  
 if **itype** = 3,  $Z^T B^{-1} Z = I$ .

If **job** = Nag\_EigVals, **z** is not referenced.

10: **pdz** – Integer Input

On entry: the stride separating row or column elements (depending on the value of **order**) in the array **z**.

Constraints:

if **job** = Nag\_DoBoth,  $\mathbf{pdz} \geq \max(1, n)$ ;  
 otherwise  $\mathbf{pdz} \geq 1$ .

11: **fail** – NagError \* Input/Output

The NAG error argument (see Section 2.7 in How to Use the NAG Library and its Documentation).

## 6 Error Indicators and Warnings

### NE\_ALLOC\_FAIL

Dynamic memory allocation failed.

See Section 2.3.1.2 in How to Use the NAG Library and its Documentation for further information.

### NE\_BAD\_PARAM

On entry, argument  $\langle value \rangle$  had an illegal value.

### NE\_CONVERGENCE

The algorithm failed to converge;  $\langle value \rangle$  off-diagonal elements of an intermediate tridiagonal form did not converge to zero.

### NE\_ENUM\_INT\_2

On entry, **job** =  $\langle value \rangle$ , **n** =  $\langle value \rangle$  and **pdz** =  $\langle value \rangle$ .

Constraint: if **job** = Nag\_DoBoth, **pdz**  $\geq \max(1, \mathbf{n})$ ;  
otherwise **pdz**  $\geq 1$ .

On entry, **job** =  $\langle value \rangle$ , **pdz** =  $\langle value \rangle$ , **n** =  $\langle value \rangle$ .

Constraint: if **job** = Nag\_DoBoth, **pdz**  $\geq \max(1, \mathbf{n})$ ;  
otherwise **pdz**  $\geq 1$ .

### NE\_INT

On entry, **itype** =  $\langle value \rangle$ .

Constraint: **itype** = 1, 2 or 3.

On entry, **n** =  $\langle value \rangle$ .

Constraint: **n**  $\geq 0$ .

On entry, **pdz** =  $\langle value \rangle$ .

Constraint: **pdz**  $> 0$ .

### NE\_INTERNAL\_ERROR

An internal error has occurred in this function. Check the function call and any array sizes. If the call is correct then please contact NAG for assistance.

An unexpected error has been triggered by this function. Please contact NAG.

See Section 2.7.6 in How to Use the NAG Library and its Documentation for further information.

### NE\_MAT\_NOT\_POS\_DEF

If **fail.errnum** = **n** +  $\langle value \rangle$ , for  $1 \leq \langle value \rangle \leq \mathbf{n}$ , then the leading minor of order  $\langle value \rangle$  of  $B$  is not positive definite. The factorization of  $B$  could not be completed and no eigenvalues or eigenvectors were computed.

### NE\_NO\_LICENCE

Your licence key may have expired or may not have been installed correctly.

See Section 2.7.5 in How to Use the NAG Library and its Documentation for further information.

## 7 Accuracy

If  $B$  is ill-conditioned with respect to inversion, then the error bounds for the computed eigenvalues and vectors may be large, although when the diagonal elements of  $B$  differ widely in magnitude the eigenvalues and eigenvectors may be less sensitive than the condition of  $B$  would suggest. See Section 4.10 of Anderson *et al.* (1999) for details of the error bounds.

The example program below illustrates the computation of approximate error bounds.

## 8 Parallelism and Performance

nag\_dspgvd (f08tcc) is threaded by NAG for parallel execution in multithreaded implementations of the NAG Library.

nag\_dspgvd (f08tcc) makes calls to BLAS and/or LAPACK routines, which may be threaded within the vendor library used by this implementation. Consult the documentation for the vendor library for further information.

Please consult the x06 Chapter Introduction for information on how to control and interrogate the OpenMP environment used within this function. Please also consult the Users' Note for your implementation for any additional implementation-specific information.

## 9 Further Comments

The total number of floating-point operations is proportional to  $n^3$ .

The complex analogue of this function is nag\_zhpgvd (f08tqc).

## 10 Example

This example finds all the eigenvalues and eigenvectors of the generalized symmetric eigenproblem  $ABz = \lambda z$ , where

$$A = \begin{pmatrix} 0.24 & 0.39 & 0.42 & -0.16 \\ 0.39 & -0.11 & 0.79 & 0.63 \\ 0.42 & 0.79 & -0.25 & 0.48 \\ -0.16 & 0.63 & 0.48 & -0.03 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 4.16 & -3.12 & 0.56 & -0.10 \\ -3.12 & 5.03 & -0.83 & 1.09 \\ 0.56 & -0.83 & 0.76 & 0.34 \\ -0.10 & 1.09 & 0.34 & 1.18 \end{pmatrix},$$

together with an estimate of the condition number of  $B$ , and approximate error bounds for the computed eigenvalues and eigenvectors.

The example program for nag\_dspgv (f08tac) illustrates solving a generalized symmetric eigenproblem of the form  $Az = \lambda Bz$ .

### 10.1 Program Text

```
/* nag_dspgvd (f08tcc) Example Program.
 *
 * NAGPRODCODE Version.
 *
 * Copyright 2016 Numerical Algorithms Group.
 *
 * Mark 26, 2016.
 */

#include <stdio.h>
#include <math.h>
#include <nag.h>
#include <nag_stdlib.h>
#include <nagf07.h>
#include <nagf08.h>
#include <nagf16.h>
#include <nagx02.h>

int main(void)
{
    /* Scalars */
    double anorm, bnorm, eps, rcond, rcondb, t1;
    Integer i, j, n;
    Integer exit_status = 0;
    /* Arrays */
    double *ap = 0, *bp = 0, *eerbnd = 0, *w = 0;
    double dummy[1];
    char nag_enum_arg[40];
```

```

/* Nag Types */
NagError fail;
Nag_OrderType order;
Nag_UploType uplo;

#ifdef NAG_COLUMN_MAJOR
#define A_UPPER(I, J) ap[J*(J-1)/2 + I - 1]
#define A_LOWER(I, J) ap[(2*n-J)*(J-1)/2 + I - 1]
#define B_UPPER(I, J) bp[J*(J-1)/2 + I - 1]
#define B_LOWER(I, J) bp[(2*n-J)*(J-1)/2 + I - 1]
    order = Nag_ColMajor;
#else
#define A_UPPER(I, J) ap[(2*n-I)*(I-1)/2 + J - 1]
#define A_LOWER(I, J) ap[I*(I-1)/2 + J - 1]
#define B_UPPER(I, J) bp[(2*n-I)*(I-1)/2 + J - 1]
#define B_LOWER(I, J) bp[I*(I-1)/2 + J - 1]
    order = Nag_RowMajor;
#endif

    INIT_FAIL(fail);

    printf("nag_dspgvd (f08tcc) Example Program Results\n\n");

    /* Skip heading in data file */
#ifdef _WIN32
    scanf_s("%*[\n]");
#else
    scanf("%*[\n]");
#endif
#ifdef _WIN32
    scanf_s("%" NAG_IFMT "%*[\n]", &n);
#else
    scanf("%" NAG_IFMT "%*[\n]", &n);
#endif
    if (n < 0) {
        printf("Invalid n\n");
        exit_status = 1;
        goto END;;
    }
#ifdef _WIN32
    scanf_s(" %39s%*[\n]", nag_enum_arg, (unsigned)_countof(nag_enum_arg));
#else
    scanf(" %39s%*[\n]", nag_enum_arg);
#endif
    /* nag_enum_name_to_value (x04nac).
     * Converts NAG enum member name to value
     */
    uplo = (Nag_UploType) nag_enum_name_to_value(nag_enum_arg);

    /* Allocate memory */
    if (!(ap = NAG_ALLOC(n * (n + 1) / 2, double)) ||
        !(bp = NAG_ALLOC(n * (n + 1) / 2, double)) ||
        !(eerbnd = NAG_ALLOC(n, double)) || !(w = NAG_ALLOC(n, double)))
    {
        printf("Allocation failure\n");
        exit_status = -1;
        goto END;
    }

    /* Read the triangular parts of the matrices A and B from data file. */
    if (uplo == Nag_Upper) {
        for (i = 1; i <= n; ++i)
#ifdef _WIN32
            for (j = i; j <= n; ++j)
                scanf_s("%lf", &A_UPPER(i, j));
#else
            for (j = i; j <= n; ++j)
                scanf("%lf", &A_UPPER(i, j));
#endif
    }
#ifdef _WIN32
    scanf_s("%*[\n]");

```

```

#else
    scanf("%*[\n]");
#endif
    for (i = 1; i <= n; ++i)
#ifdef _WIN32
        for (j = i; j <= n; ++j)
            scanf_s("%lf", &B_UPPER(i, j));
#else
        for (j = i; j <= n; ++j)
            scanf("%lf", &B_UPPER(i, j));
#endif
    }
    else if (uplo == Nag_Lower) {
        for (i = 1; i <= n; ++i)
#ifdef _WIN32
            for (j = 1; j <= i; ++j)
                scanf_s("%lf", &A_LOWER(i, j));
#else
            for (j = 1; j <= i; ++j)
                scanf("%lf", &A_LOWER(i, j));
#endif
#ifdef _WIN32
            scanf_s("%*[\n]");
#else
            scanf("%*[\n]");
#endif
        for (i = 1; i <= n; ++i)
#ifdef _WIN32
            for (j = 1; j <= i; ++j)
                scanf_s("%lf", &B_LOWER(i, j));
#else
            for (j = 1; j <= i; ++j)
                scanf("%lf", &B_LOWER(i, j));
#endif
    }
#ifdef _WIN32
    scanf_s("%*[\n]");
#else
    scanf("%*[\n]");
#endif

/* Compute the one-norms of the symmetric matrices A and B
 * using nag_dsp_norm (f16rdc).
 */
nag_dsp_norm(order, Nag_OneNorm, uplo, n, ap, &anorm, &fail);
nag_dsp_norm(order, Nag_OneNorm, uplo, n, bp, &bnorm, &fail);
if (fail.code != NE_NOERROR) {
    printf("Error from nag_dsp_norm (f16rdc).\n%s\n", fail.message);
    exit_status = 1;
    goto END;
}

/* Solve the generalized symmetric eigenvalue problem A*B*x = lambda*x
 * using nag_dspgvd (f08tcc).
 * In the following call the 9th argument is set to n rather than 1 to
 * avoid an incorrect error message in some vendor versions of LAPACK.
 */
nag_dspgvd(order, 2, Nag_EigVals, uplo, n, ap, bp, w, dummy, n, &fail);
if (fail.code != NE_NOERROR) {
    printf("Error from nag_dspgvd (f08tcc).\n%s\n", fail.message);
    exit_status = 1;
    goto END;
}

/* Print eigensolution */
printf("Eigenvalues\n");
for (j = 0; j < n; ++j)
    printf(" %11.4f%s", w[j], j % 6 == 5 ? "\n" : "");
printf("\n");

/* Estimate the reciprocal condition number of the Cholesky factor of B.

```

```

    * nag_dtpcon (f07ugc).
    * Note that: cond(B) = 1/(rcond*rcond).
    */
nag_dtpcon(order, Nag_OneNorm, uplo, Nag_NonUnitDiag, n, bp, &rcond, &fail);
if (fail.code != NE_NOERROR) {
    printf("Error from nag_dtpcon (f07ugc).\n%s\n", fail.message);
    exit_status = 1;
    goto END;
}

/* Print the reciprocal condition number of B */
rcondb = rcond * rcond;
printf("\nEstimate of reciprocal condition number for B\n      %11.1e\n",
      rcondb);

/* Get the machine precision, using nag_machine_precision (x02ajc) */
eps = nag_machine_precision;
if (rcond < eps) {
    printf("\nB is very ill-conditioned, error estimates have not been"
          " computed\n");
    goto END;
}

t1 = anorm * bnorm;
for (i = 0; i < n; ++i)
    eerbnd[i] = eps * (t1 + fabs(w[i]) / rcondb);

/* Print the approximate error bounds for the eigenvalues */
printf("\nError estimates for the eigenvalues\n      ");
for (i = 0; i < n; ++i)
    printf(" %11.1e%s", eerbnd[i], i % 6 == 5 ? "\n" : "");
printf("\n");

END:
NAG_FREE(ap);
NAG_FREE(bp);
NAG_FREE(eerbnd);
NAG_FREE(w);

return exit_status;
}

```

## 10.2 Program Data

nag\_dspgvd (f08tcc) Example Program Data

```

4                               : n

Nag_Upper                      : uplo

0.24   0.39   0.42  -0.16
      -0.11   0.79   0.63
           -0.25   0.48
                -0.03 : matrix A

4.16  -3.12   0.56  -0.10
      5.03  -0.83   1.09
           0.76   0.34
                1.18 : matrix B

```



**10.3 Program Results**

nag\_dspgvd (f08tcc) Example Program Results

Eigenvalues

-3.5411	-0.3347	0.2983	2.2544
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Estimate of reciprocal condition number for B  
5.8e-03

Error estimates for the eigenvalues

7.0e-14	8.6e-15	7.9e-15	4.6e-14
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