

# NAG Library Function Document

## nag\_ztrsyl (f08qvc)

### 1 Purpose

nag\_ztrsyl (f08qvc) solves the complex triangular Sylvester matrix equation.

### 2 Specification

```
#include <nag.h>
#include <nagf08.h>

void nag_ztrsyl (Nag_OrderType order, Nag_TransType trana,
                 Nag_TransType tranb, Nag_SignType sign, Integer m, Integer n,
                 const Complex a[], Integer pda, const Complex b[], Integer pdb,
                 Complex c[], Integer pdc, double *scal, NagError *fail)
```

### 3 Description

nag\_ztrsyl (f08qvc) solves the complex Sylvester matrix equation

$$\text{op}(A)X \pm X\text{op}(B) = \alpha C,$$

where  $\text{op}(A) = A$  or  $A^H$ , and the matrices  $A$  and  $B$  are upper triangular;  $\alpha$  is a scale factor ( $\leq 1$ ) determined by the function to avoid overflow in  $X$ ;  $A$  is  $m$  by  $m$  and  $B$  is  $n$  by  $n$  while the right-hand side matrix  $C$  and the solution matrix  $X$  are both  $m$  by  $n$ . The matrix  $X$  is obtained by a straightforward process of back-substitution (see Golub and Van Loan (1996)).

Note that the equation has a unique solution if and only if  $\alpha_i \pm \beta_j \neq 0$ , where  $\{\alpha_i\}$  and  $\{\beta_j\}$  are the eigenvalues of  $A$  and  $B$  respectively and the sign (+ or  $-$ ) is the same as that used in the equation to be solved.

### 4 References

Golub G H and Van Loan C F (1996) *Matrix Computations* (3rd Edition) Johns Hopkins University Press, Baltimore

Higham N J (1992) Perturbation theory and backward error for  $AX - XB = C$  *Numerical Analysis Report* University of Manchester

### 5 Arguments

1: **order** – Nag\_OrderType *Input*

*On entry:* the **order** argument specifies the two-dimensional storage scheme being used, i.e., row-major ordering or column-major ordering. C language defined storage is specified by **order** = Nag\_RowMajor. See Section 2.3.1.3 in How to Use the NAG Library and its Documentation for a more detailed explanation of the use of this argument.

*Constraint:* **order** = Nag\_RowMajor or Nag\_ColMajor.

2: **trana** – Nag\_TransType *Input*

*On entry:* specifies the option  $\text{op}(A)$ .

**trana** = Nag\_NoTrans  
 $\text{op}(A) = A$ .

**trana** = Nag\_ConjTrans

$$\text{op}(A) = A^H.$$

*Constraint:* **trana** = Nag\_NoTrans or Nag\_ConjTrans.

3: **tranb** – Nag\_TransType

*Input*

*On entry:* specifies the option  $\text{op}(B)$ .

**tranb** = Nag\_NoTrans

$$\text{op}(B) = B.$$

**tranb** = Nag\_ConjTrans

$$\text{op}(B) = B^H.$$

*Constraint:* **tranb** = Nag\_NoTrans or Nag\_ConjTrans.

4: **sign** – Nag\_SignType

*Input*

*On entry:* indicates the form of the Sylvester equation.

**sign** = Nag\_Plus

$$\text{The equation is of the form } \text{op}(A)X + X\text{op}(B) = \alpha C.$$

**sign** = Nag\_Minus

$$\text{The equation is of the form } \text{op}(A)X - X\text{op}(B) = \alpha C.$$

*Constraint:* **sign** = Nag\_Plus or Nag\_Minus.

5: **m** – Integer

*Input*

*On entry:*  $m$ , the order of the matrix  $A$ , and the number of rows in the matrices  $X$  and  $C$ .

*Constraint:*  $\mathbf{m} \geq 0$ .

6: **n** – Integer

*Input*

*On entry:*  $n$ , the order of the matrix  $B$ , and the number of columns in the matrices  $X$  and  $C$ .

*Constraint:*  $\mathbf{n} \geq 0$ .

7: **a**[ $dim$ ] – const Complex

*Input*

**Note:** the dimension,  $dim$ , of the array **a** must be at least  $\max(1, \mathbf{pda} \times \mathbf{m})$ .

The  $(i, j)$ th element of the matrix  $A$  is stored in

$$\mathbf{a}[(j-1) \times \mathbf{pda} + i - 1] \text{ when } \mathbf{order} = \text{Nag\_ColMajor};$$

$$\mathbf{a}[(i-1) \times \mathbf{pda} + j - 1] \text{ when } \mathbf{order} = \text{Nag\_RowMajor}.$$

*On entry:* the  $m$  by  $m$  upper triangular matrix  $A$ .

8: **pda** – Integer

*Input*

*On entry:* the stride separating row or column elements (depending on the value of **order**) in the array **a**.

*Constraint:*  $\mathbf{pda} \geq \max(1, \mathbf{m})$ .

9: **b**[ $dim$ ] – const Complex

*Input*

**Note:** the dimension,  $dim$ , of the array **b** must be at least  $\max(1, \mathbf{pdb} \times \mathbf{n})$ .

The  $(i, j)$ th element of the matrix  $B$  is stored in

$$\mathbf{b}[(j-1) \times \mathbf{pdb} + i - 1] \text{ when } \mathbf{order} = \text{Nag\_ColMajor};$$

$$\mathbf{b}[(i-1) \times \mathbf{pdb} + j - 1] \text{ when } \mathbf{order} = \text{Nag\_RowMajor}.$$

*On entry:* the  $n$  by  $n$  upper triangular matrix  $B$ .

- 10: **pdb** – Integer *Input*  
*On entry:* the stride separating row or column elements (depending on the value of **order**) in the array **b**.  
*Constraint:* **pdb**  $\geq \max(1, n)$ .
- 11: **c**[*dim*] – Complex *Input/Output*  
**Note:** the dimension, *dim*, of the array **c** must be at least  
 $\max(1, \mathbf{pdc} \times n)$  when **order** = Nag\_ColMajor;  
 $\max(1, m \times \mathbf{pdc})$  when **order** = Nag\_RowMajor.  
The (*i*, *j*)th element of the matrix *C* is stored in  
**c**[(*j* – 1)  $\times$  **pdc** + *i* – 1] when **order** = Nag\_ColMajor;  
**c**[(*i* – 1)  $\times$  **pdc** + *j* – 1] when **order** = Nag\_RowMajor.  
*On entry:* the *m* by *n* right-hand side matrix *C*.  
*On exit:* **c** is overwritten by the solution matrix *X*.
- 12: **pdc** – Integer *Input*  
*On entry:* the stride separating row or column elements (depending on the value of **order**) in the array **c**.  
*Constraints:*  
if **order** = Nag\_ColMajor, **pdc**  $\geq \max(1, m)$ ;  
if **order** = Nag\_RowMajor, **pdc**  $\geq \max(1, n)$ .
- 13: **scal** – double \* *Output*  
*On exit:* the value of the scale factor  $\alpha$ .
- 14: **fail** – NagError \* *Input/Output*  
The NAG error argument (see Section 2.7 in How to Use the NAG Library and its Documentation).

## 6 Error Indicators and Warnings

### NE\_ALLOC\_FAIL

Dynamic memory allocation failed.

See Section 2.3.1.2 in How to Use the NAG Library and its Documentation for further information.

### NE\_BAD\_PARAM

On entry, argument  $\langle value \rangle$  had an illegal value.

### NE\_INT

On entry, **m** =  $\langle value \rangle$ .

Constraint: **m**  $\geq 0$ .

On entry, **n** =  $\langle value \rangle$ .

Constraint: **n**  $\geq 0$ .

On entry, **pda** =  $\langle value \rangle$ .

Constraint: **pda**  $> 0$ .

On entry, **pdb** =  $\langle value \rangle$ .

Constraint: **pdb**  $> 0$ .

On entry, **pdc** =  $\langle value \rangle$ .  
 Constraint: **pdc** > 0.

## NE\_INT\_2

On entry, **pda** =  $\langle value \rangle$  and **m** =  $\langle value \rangle$ .  
 Constraint: **pda**  $\geq$  max(1, **m**).

On entry, **pdb** =  $\langle value \rangle$  and **n** =  $\langle value \rangle$ .  
 Constraint: **pdb**  $\geq$  max(1, **n**).

On entry, **pdc** =  $\langle value \rangle$  and **m** =  $\langle value \rangle$ .  
 Constraint: **pdc**  $\geq$  max(1, **m**).

On entry, **pdc** =  $\langle value \rangle$  and **n** =  $\langle value \rangle$ .  
 Constraint: **pdc**  $\geq$  max(1, **n**).

## NE\_INTERNAL\_ERROR

An internal error has occurred in this function. Check the function call and any array sizes. If the call is correct then please contact NAG for assistance.

An unexpected error has been triggered by this function. Please contact NAG.  
 See Section 2.7.6 in How to Use the NAG Library and its Documentation for further information.

## NE\_NO\_LICENCE

Your licence key may have expired or may not have been installed correctly.  
 See Section 2.7.5 in How to Use the NAG Library and its Documentation for further information.

## NE\_PERTURBED

$A$  and  $B$  have common or close eigenvalues, perturbed values of which were used to solve the equation.

## 7 Accuracy

Consider the equation  $AX - XB = C$ . (To apply the remarks to the equation  $AX + XB = C$ , simply replace  $B$  by  $-B$ .)

Let  $\tilde{X}$  be the computed solution and  $R$  the residual matrix:

$$R = C - (A\tilde{X} - \tilde{X}B).$$

Then the residual is always small:

$$\|R\|_F = O(\epsilon)(\|A\|_F + \|B\|_F)\|\tilde{X}\|_F.$$

However,  $\tilde{X}$  is **not** necessarily the exact solution of a slightly perturbed equation; in other words, the solution is not backwards stable.

For the forward error, the following bound holds:

$$\|\tilde{X} - X\|_F \leq \frac{\|R\|_F}{sep(A, B)}$$

but this may be a considerable over estimate. See Golub and Van Loan (1996) for a definition of  $sep(A, B)$ , and Higham (1992) for further details.

These remarks also apply to the solution of a general Sylvester equation, as described in Section 9.

## 8 Parallelism and Performance

nag\_ztrsyl (f08qvc) makes calls to BLAS and/or LAPACK routines, which may be threaded within the vendor library used by this implementation. Consult the documentation for the vendor library for further information.

Please consult the x06 Chapter Introduction for information on how to control and interrogate the OpenMP environment used within this function. Please also consult the Users' Note for your implementation for any additional implementation-specific information.

## 9 Further Comments

The total number of real floating-point operations is approximately  $4mn(m+n)$ .

To solve the **general** complex Sylvester equation

$$AX \pm XB = C$$

where  $A$  and  $B$  are general matrices,  $A$  and  $B$  must first be reduced to Schur form :

$$A = Q_1 \tilde{A} Q_1^H \quad \text{and} \quad B = Q_2 \tilde{B} Q_2^H$$

where  $\tilde{A}$  and  $\tilde{B}$  are upper triangular and  $Q_1$  and  $Q_2$  are unitary. The original equation may then be transformed to:

$$\tilde{A} \tilde{X} \pm \tilde{X} \tilde{B} = \tilde{C}$$

where  $\tilde{X} = Q_1^H X Q_2$  and  $\tilde{C} = Q_1^H C Q_2$ .  $\tilde{C}$  may be computed by matrix multiplication; nag\_ztrsyl (f08qvc) may be used to solve the transformed equation; and the solution to the original equation can be obtained as  $X = Q_1 \tilde{X} Q_2^H$ .

The real analogue of this function is nag\_dtrsyl (f08qhc).

## 10 Example

This example solves the Sylvester equation  $AX + XB = C$ , where

$$A = \begin{pmatrix} -6.00 - 7.00i & 0.36 - 0.36i & -0.19 + 0.48i & 0.88 - 0.25i \\ 0.00 + 0.00i & -5.00 + 2.00i & -0.03 - 0.72i & -0.23 + 0.13i \\ 0.00 + 0.00i & 0.00 + 0.00i & 8.00 - 1.00i & 0.94 + 0.53i \\ 0.00 + 0.00i & 0.00 + 0.00i & 0.00 + 0.00i & 3.00 - 4.00i \end{pmatrix},$$

$$B = \begin{pmatrix} 0.50 - 0.20i & -0.29 - 0.16i & -0.37 + 0.84i & -0.55 + 0.73i \\ 0.00 + 0.00i & -0.40 + 0.90i & 0.06 + 0.22i & -0.43 + 0.17i \\ 0.00 + 0.00i & 0.00 + 0.00i & -0.90 - 0.10i & -0.89 - 0.42i \\ 0.00 + 0.00i & 0.00 + 0.00i & 0.00 + 0.00i & 0.30 - 0.70i \end{pmatrix}$$

and

$$C = \begin{pmatrix} 0.63 + 0.35i & 0.45 - 0.56i & 0.08 - 0.14i & -0.17 - 0.23i \\ -0.17 + 0.09i & -0.07 - 0.31i & 0.27 - 0.54i & 0.35 + 1.21i \\ -0.93 - 0.44i & -0.33 - 0.35i & 0.41 - 0.03i & 0.57 + 0.84i \\ 0.54 + 0.25i & -0.62 - 0.05i & -0.52 - 0.13i & 0.11 - 0.08i \end{pmatrix}.$$

### 10.1 Program Text

```
/* nag_ztrsyl (f08qvc) Example Program.
*
* NAGPRODCODE Version.
*
* Copyright 2016 Numerical Algorithms Group.
*
* Mark 26, 2016.
*/
```

```

#include <stdio.h>
#include <math.h>
#include <nag.h>
#include <nag_stdlib.h>
#include <nagf08.h>
#include <nagf16.h>
#include <nagx04.h>
#include <nagx02.h>

int main(void)
{
    /* Scalars */
    Complex alpha, beta;
    Integer i, j, m, n, pda, pdb, pdc, pdd, pde, pdf;
    Integer exit_status = 0;
    double norm, scale;
    NagError fail;
    Nag_OrderType order;
    Nag_SignType sign = Nag_Minus;
    /* Arrays */
    Complex *a = 0, *b = 0, *c = 0, *d = 0, *e = 0, *f = 0;

#ifdef NAG_COLUMN_MAJOR
#define A(I, J) a[(J-1)*pda + I - 1]
#define B(I, J) b[(J-1)*pdb + I - 1]
#define C(I, J) c[(J-1)*pdc + I - 1]
#define D(I, J) d[(J-1)*pdd + I - 1]
#define E(I, J) e[(J-1)*pde + I - 1]
#define F(I, J) f[(J-1)*pdf + I - 1]
    order = Nag_ColMajor;
#else
#define A(I, J) a[(I-1)*pda + J - 1]
#define B(I, J) b[(I-1)*pdb + J - 1]
#define C(I, J) c[(I-1)*pdc + J - 1]
#define D(I, J) d[(I-1)*pdd + J - 1]
#define E(I, J) e[(I-1)*pde + J - 1]
#define F(I, J) f[(I-1)*pdf + J - 1]
    order = Nag_RowMajor;
#endif

    INIT_FAIL(fail);

    printf("nag_ztrsyl (f08qvc) Example Program Results\n\n");

    /* Skip heading in data file */
#ifdef _WIN32
    scanf_s("%*[\n] ");
#else
    scanf("%*[\n] ");
#endif
#ifdef _WIN32
    scanf_s("%" NAG_IFMT "%" NAG_IFMT "%*[\n] ", &m, &n);
#else
    scanf("%" NAG_IFMT "%" NAG_IFMT "%*[\n] ", &m, &n);
#endif
    pda = m;
    pdb = n;
#ifdef NAG_COLUMN_MAJOR
    pdc = m;
    pdd = m;
    pde = m;
    pdf = m;
#else
    pdc = n;
    pdd = n;
    pde = n;
    pdf = n;
#endif

    /* Allocate memory */
    if (!(a = NAG_ALLOC(m * m, Complex)) ||

```

```

        !(b = NAG_ALLOC(n * m, Complex)) ||
        !(c = NAG_ALLOC(m * n, Complex)) ||
        !(d = NAG_ALLOC(m * n, Complex)) ||
        !(e = NAG_ALLOC(m * n, Complex)) || !(f = NAG_ALLOC(m * n, Complex))
    )
{
    printf("Allocation failure\n");
    exit_status = -1;
    goto END;
}

/* Read A, B and C from data file */
for (i = 1; i <= m; ++i) {
    for (j = 1; j <= m; ++j)
#ifdef _WIN32
        scanf_s(" ( %lf , %lf ) ", &A(i, j).re, &A(i, j).im);
    #else
        scanf(" ( %lf , %lf ) ", &A(i, j).re, &A(i, j).im);
    #endif
}
#ifdef _WIN32
    scanf_s("%*[\n] ");
#else
    scanf("%*[\n] ");
#endif
    for (i = 1; i <= n; ++i) {
        for (j = 1; j <= n; ++j)
#ifdef _WIN32
            scanf_s(" ( %lf , %lf ) ", &B(i, j).re, &B(i, j).im);
        #else
            scanf(" ( %lf , %lf ) ", &B(i, j).re, &B(i, j).im);
        #endif
    }
#ifdef _WIN32
    scanf_s("%*[\n] ");
#else
    scanf("%*[\n] ");
#endif
    for (i = 1; i <= m; ++i) {
        for (j = 1; j <= n; ++j)
#ifdef _WIN32
            scanf_s(" ( %lf , %lf ) ", &C(i, j).re, &C(i, j).im);
        #else
            scanf(" ( %lf , %lf ) ", &C(i, j).re, &C(i, j).im);
        #endif
    }
#ifdef _WIN32
    scanf_s("%*[\n] ");
#else
    scanf("%*[\n] ");
#endif

/* Copy C into F */
for (i = 1; i <= m; i++) {
    for (j = 1; j <= m; j++) {
        F(i, j).re = C(i, j).re;
        F(i, j).im = C(i, j).im;
    }
}

/* nag_gen_complx_mat_print_comp (x04dbc): Print matrix C */
fflush(stdout);
nag_gen_complx_mat_print_comp(order, Nag_GeneralMatrix, Nag_NonUnitDiag, m,
                               n, c, pdc, Nag_BracketForm, "%7.4f",
                               "Matrix C", Nag_IntegerLabels, 0,
                               Nag_IntegerLabels, 0, 80, 0, 0, &fail);

printf("\n");
if (fail.code != NE_NOERROR) {
    printf("Error from nag_gen_complx_mat_print_comp (x04dbc).\n%s\n",
           fail.message);
    exit_status = 1;
}

```

```

    goto END;
}

/* Reorder the Schur factorization T */
/* nag_ztrsyl (f08qvc).
 * Solve complex Sylvester matrix equation  $AX + XB = C$ , A
 * and B are upper triangular or conjugate-transposes
 */
nag_ztrsyl(order, Nag_NoTrans, Nag_NoTrans, sign, m, n, a, pda,
           b, pdb, c, pdc, &scale, &fail);
if (fail.code != NE_NOERROR) {
    printf("Error from nag_ztrsyl (f08qvc).\n%s\n", fail.message);
    exit_status = 1;
    goto END;
}

/* nag_zgemm (f16zac): Compute  $aC - (A*X + X*B*sign)$  from the solution */
/* and store in matrix E */
alpha.re = 1.0;
alpha.im = 0.0;
beta.re = 0.0;
beta.im = 0.0;
nag_zgemm(order, Nag_NoTrans, Nag_NoTrans, m, n, m, alpha, a, pda,
           c, pdc, beta, d, pdd, &fail);
if (fail.code != NE_NOERROR) {
    printf("Error from nag_zgemm (f16zac).\n%s\n", fail.message);
    exit_status = 1;
    goto END;
}
if (sign == Nag_Minus)
    alpha.re = -1.0;
else
    alpha.re = 1.0;
beta.re = 1.0;
nag_zgemm(order, Nag_NoTrans, Nag_NoTrans, m, n, n, alpha, c, pdc,
           b, pdb, beta, d, pdd, &fail);
if (fail.code != NE_NOERROR) {
    printf("Error from nag_zgemm (f16zac).\n%s\n", fail.message);
    exit_status = 1;
    goto END;
}
for (i = 1; i <= m; i++) {
    for (j = 1; j <= n; j++) {
        E(i, j).re = scale * F(i, j).re - D(i, j).re;
        E(i, j).im = scale * F(i, j).im - D(i, j).im;
    }
}

/* nag_zge_norm (f16uac): Find norm of matrix E and print warning if */
/* it is too large */
nag_zge_norm(order, Nag_OneNorm, m, n, e, pde, &norm, &fail);
if (fail.code != NE_NOERROR) {
    printf("Error from nag_dge_norm (f16rac).\n%s\n", fail.message);
    exit_status = 1;
    goto END;
}
if (norm > pow(x02ajc(), 0.8)) {
    printf("%s\n%s\n",
           "Norm of  $aC - (A*X + X*B*sign)$  is much greater than 0.",
           "Schur factorization has failed.");
}
else {
    printf(" SCALE = %11.2e\n", scale);
}

END:
NAG_FREE(a);
NAG_FREE(b);
NAG_FREE(c);
NAG_FREE(d);

```



```

NAG_FREE(e);
NAG_FREE(f);

return exit_status;
}

```

## 10.2 Program Data

```

nag_ztrsyl (f08qvc) Example Program Data
  4  4                                     :Values of M and N
(-6.00,-7.00) ( 0.36,-0.36) (-0.19, 0.48) ( 0.88,-0.25)
( 0.00, 0.00) (-5.00, 2.00) (-0.03,-0.72) (-0.23, 0.13)
( 0.00, 0.00) ( 0.00, 0.00) ( 8.00,-1.00) ( 0.94, 0.53)
( 0.00, 0.00) ( 0.00, 0.00) ( 0.00, 0.00) ( 3.00,-4.00) :End of matrix A
( 0.50,-0.20) (-0.29,-0.16) (-0.37, 0.84) (-0.55, 0.73)
( 0.00, 0.00) (-0.40, 0.90) ( 0.06, 0.22) (-0.43, 0.17)
( 0.00, 0.00) ( 0.00, 0.00) (-0.90,-0.10) (-0.89,-0.42)
( 0.00, 0.00) ( 0.00, 0.00) ( 0.00, 0.00) ( 0.30,-0.70) :End of matrix B
( 0.63, 0.35) ( 0.45,-0.56) ( 0.08,-0.14) (-0.17,-0.23)
(-0.17, 0.09) (-0.07,-0.31) ( 0.27,-0.54) ( 0.35, 1.21)
(-0.93,-0.44) (-0.33,-0.35) ( 0.41,-0.03) ( 0.57, 0.84)
( 0.54, 0.25) (-0.62,-0.05) (-0.52,-0.13) ( 0.11,-0.08) :End of matrix C

```

## 10.3 Program Results

```

nag_ztrsyl (f08qvc) Example Program Results

Matrix C
          1          2          3          4
1 ( 0.6300, 0.3500) ( 0.4500,-0.5600) ( 0.0800,-0.1400) (-0.1700,-0.2300)
2 (-0.1700, 0.0900) (-0.0700,-0.3100) ( 0.2700,-0.5400) ( 0.3500, 1.2100)
3 (-0.9300,-0.4400) (-0.3300,-0.3500) ( 0.4100,-0.0300) ( 0.5700, 0.8400)
4 ( 0.5400, 0.2500) (-0.6200,-0.0500) (-0.5200,-0.1300) ( 0.1100,-0.0800)

SCALE = 1.00e+00

```

---