NAG Library Function Document

nag_pde_bs_1d_analytic (d03ndc)

1 Purpose

nag_pde_bs_1d_analytic (d03ndc) computes an analytic solution to the Black–Scholes equation for a certain set of option types.

2 Specification

```
#include <nag.h>
#include <nagd03.h>
void nag_pde_bs_1d_analytic (Nag_OptionType kopt, double x, double s,
     double t, double tmat, const Nag_Boolean tdpar[], const double r[],
     const double q[], const double sigma[], double *f, double *theta,
     double *delta, double *gamma, double *lambda, double *rho,
     NagError *fail)
```

3 Description

nag_pde_bs_1d_analytic (d03ndc) computes an analytic solution to the Black–Scholes equation (see Hull (1989) and Wilmott *et al.* (1995))

$$\frac{\partial f}{\partial t} + (r-q)S\frac{\partial f}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 f}{\partial S^2} = rf \tag{1}$$

$$S_{\min} < S < S_{\max}, \quad t_{\min} < t < t_{\max}, \tag{2}$$

for the value f of a European put or call option, or an American call option with zero dividend q. In equation (1) t is time, S is the stock price, X is the exercise price, r is the risk free interest rate, q is the continuous dividend, and σ is the stock volatility. The parameter r, q and σ may be either constant, or functions of time. In the latter case their average instantaneous values over the remaining life of the option should be provided to nag_pde_bs_1d_analytic (d03ndc). An auxiliary function nag_pde_bs_1d_means (d03nec) is available to compute such averages from values at a set of discrete times. Equation (1) is subject to different boundary conditions depending on the type of option. For a call option the boundary condition is

$$f(S, t = t_{\text{mat}}) = \max(0, S - X)$$

where t_{mat} is the maturity time of the option. For a put option the equation (1) is subject to

$$f(S, t = t_{\text{mat}}) = \max(0, X - S).$$

nag_pde_bs_1d_analytic (d03ndc) also returns values of the Greeks

$$\Theta = \frac{\partial f}{\partial t}, \quad \Delta = \frac{\partial f}{\partial x}, \quad \Gamma = \frac{\partial^2 f}{\partial x^2}, \quad \Lambda = \frac{\partial f}{\partial \sigma}, \quad \rho = \frac{\partial f}{\partial r}.$$

nag_bsm_greeks (s30abc) also computes the European option price given by the Black–Scholes–Merton formula together with a more comprehensive set of sensitivities (Greeks).

Further details of the analytic solution returned are given in Section 9.1.

4 References

Hull J (1989) *Options, Futures and Other Derivative Securities* Prentice–Hall

Wilmott P, Howison S and Dewynne J (1995) *The Mathematics of Financial Derivatives* Cambridge University Press

5 Arguments

1: **kopt** – Nag_OptionType *Input*

On entry: specifies the kind of option to be valued:

kopt = Nag_EuropeanCall
A European call option.

kopt = Nag_AmericanCall
An American call option.

kopt = Nag_EuropeanPut
A European put option.

Constraints:

kopt = Nag_EuropeanCall, Nag_AmericanCall or Nag_EuropeanPut;
if $q \neq 0$, **kopt** $\neq$ Nag_AmericanCall.

2: **x** – double *Input*

On entry: the exercise price X.

Constraint: **x** ≥ 0.0.

3: **s** – double *Input*

On entry: the stock price at which the option value and the Greeks should be evaluated.

Constraint: **s** ≥ 0.0.

4: **t** – double *Input*

On entry: the time at which the option value and the Greeks should be evaluated.

Constraint: **t** ≥ 0.0.

5: **tmat** – double *Input*

On entry: the maturity time of the option.

Constraint: **tmat** $\geq$ **t**.

6: **tdpar**[**3**] – const Nag_Boolean *Input*

On entry: specifies whether or not various arguments are time-dependent. More precisely, r is time-dependent if **tdpar**[0] = Nag_TRUE and constant otherwise. Similarly, **tdpar**[1] specifies whether q is time-dependent and **tdpar**[2] specifies whether σ is time-dependent.

7: **r**[dim] – const double *Input*

Note: the dimension, *dim*, of the array **r** must be at least

3 when **tdpar**[0] = Nag_TRUE;
1 otherwise.

On entry: if **tdpar**[0] = Nag_FALSE then **r**[0] must contain the constant value of r. The remaining elements need not be set.

If **tdpar**[0] = Nag_TRUE then **r**[0] must contain the value of r at time **t** and **r**[1] must contain its average instantaneous value over the remaining life of the option:

$$\hat{r} = \int_{\mathbf{t}}^{\mathbf{tmat}} r(\zeta)\, d\zeta.$$

The auxiliary function nag_pde_bs_1d_means (d03nec) may be used to construct **r** from a set of values of r at discrete times.

8: **q**[*dim*] – const double *Input*

Note: the dimension, *dim*, of the array **q** must be at least

3 when **tdpar**[1] = Nag_TRUE;
1 otherwise.

On entry: if **tdpar**[1] = Nag_FALSE then **q**[0] must contain the constant value of q. The remaining elements need not be set.

If **tdpar**[1] = Nag_TRUE then **q**[0] must contain the constant value of q and **q**[1] must contain its average instantaneous value over the remaining life of the option:

$$\hat{q} = \int_{\mathbf{t}}^{\mathbf{tmat}} q(\zeta)\, d\zeta.$$

The auxiliary function nag_pde_bs_1d_means (d03nec) may be used to construct **q** from a set of values of q at discrete times.

9: **sigma**[*dim*] – const double *Input*

Note: the dimension, *dim*, of the array **sigma** must be at least

3 when **tdpar**[2] = Nag_TRUE;
1 otherwise.

On entry: if **tdpar**[2] = Nag_FALSE then **sigma**[0] must contain the constant value of σ. The remaining elements need not be set.

If **tdpar**[2] = Nag_TRUE then **sigma**[0] must contain the value of σ at time **t**, **sigma**[1] the average instantaneous value $\hat{\sigma}$, and **sigma**[2] the second-order average $\bar{\sigma}$, where:

$$\hat{\sigma} = \int_{\mathbf{t}}^{\mathbf{tmat}} \sigma(\zeta)\, d\zeta,$$

$$\bar{\sigma} = \left(\int_{\mathbf{t}}^{\mathbf{tmat}} \sigma^2(\zeta)\, d\zeta \right)^{1/2}.$$

The auxiliary function nag_pde_bs_1d_means (d03nec) may be used to compute **sigma** from a set of values at discrete times.

Constraints:

if **tdpar**[2] = Nag_FALSE, **sigma**[0] > 0.0;
if **tdpar**[2] = Nag_TRUE, **sigma**[$i-1$] > 0.0, for $i = 1, 2, 3$.

10: **f** – double * *Output*

On exit: the value f of the option at the stock price **s** and time **t**.

11: **theta** – double * *Output*
12: **delta** – double * *Output*
13: **gamma** – double * *Output*
14: **lambda** – double * *Output*
15: **rho** – double * *Output*

On exit: the values of various Greeks at the stock price **s** and time **t**, as follows:

$$\mathbf{theta} = \Theta = \frac{\partial f}{\partial t}, \qquad \mathbf{delta} = \Delta = \frac{\partial f}{\partial \mathbf{s}}, \quad \mathbf{gamma} = \Gamma = \frac{\partial^2 f}{\partial \mathbf{s}^2},$$

$$\mathbf{lambda} = \Lambda = \frac{\partial f}{\partial \sigma}, \quad \mathbf{rho} = \rho = \frac{\partial f}{\partial r}.$$

16: **fail** – NagError * *Input/Output*

The NAG error argument (see Section 2.7 in How to Use the NAG Library and its Documentation).

6 Error Indicators and Warnings

NE_ALLOC_FAIL

Dynamic memory allocation failed.
See Section 2.3.1.2 in How to Use the NAG Library and its Documentation for further information.

NE_BAD_PARAM

On entry, argument ⟨*value*⟩ had an illegal value.

NE_INCOMPAT_PARAM

On entry, **q**[0] is not equal to 0.0 with American call option. **q**[0] = ⟨*value*⟩.

On entry, **sigma**[$I-1$] = ⟨*value*⟩ and I = ⟨*value*⟩.
Constraint: **sigma**[$I-1$] > 0.0.

NE_INTERNAL_ERROR

An internal error has occurred in this function. Check the function call and any array sizes. If the call is correct then please contact NAG for assistance.

An unexpected error has been triggered by this function. Please contact NAG.
See Section 2.7.6 in How to Use the NAG Library and its Documentation for further information.

NE_NO_LICENCE

Your licence key may have expired or may not have been installed correctly.
See Section 2.7.5 in How to Use the NAG Library and its Documentation for further information.

NE_REAL

On entry, **s** = ⟨*value*⟩.
Constraint: **s** ≥ 0.0.

On entry, **t** = ⟨*value*⟩.
Constraint: **t** ≥ 0.0.

On entry, **x** = ⟨*value*⟩.
Constraint: **x** ≥ 0.0.

NE_REAL_2

On entry, **tmat** = ⟨*value*⟩ and **t** = ⟨*value*⟩.
Constraint: **tmat** $\geq$ **t**.

7 Accuracy

Given accurate values of **r**, **q** and **sigma** no further approximations are made in the evaluation of the Black–Scholes analytic formulae, and the results should therefore be within machine accuracy. The values of **r**, **q** and **sigma** returned from nag_pde_bs_1d_means (d03nec) are exact for polynomials of degree up to 3.

8 Parallelism and Performance

nag_pde_bs_1d_analytic (d03ndc) is not threaded in any implementation.

9 Further Comments

9.1 Algorithmic Details

The Black–Scholes analytic formulae are used to compute the solution. For a European call option these are as follows:

$$f = Se^{-\hat{q}(T-t)}N(d_1) - Xe^{-\hat{r}(T-t)}N(d_2)$$

where

$$d_1 = \frac{\log(S/X) + \left(\hat{r} - \hat{q} + \bar{\sigma}^2/2\right)(T-t)}{\bar{\sigma}\sqrt{T-t}},$$

$$d_2 = \frac{\log(S/X) + \left(\hat{r} - \hat{q} - \bar{\sigma}^2/2\right)(T-t)}{\bar{\sigma}\sqrt{T-t}} = d_1 - \bar{\sigma}\sqrt{T-t},$$

$N(x)$ is the cumulative Normal distribution function and $N'(x)$ is its derivative

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\zeta^2/2}d\zeta,$$

$$N'(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}.$$

The functions $\hat{q}$, $\hat{r}$, $\hat{\sigma}$ and $\bar{\sigma}$ are average values of q, r and σ over the time to maturity:

$$\hat{q} = \frac{1}{T-t}\int_t^T q(\zeta)d\zeta,$$

$$\hat{r} = \frac{1}{T-t}\int_t^T r(\zeta)d\zeta,$$

$$\hat{\sigma} = \frac{1}{T-t}\int_t^T \sigma(\zeta)d\zeta,$$

$$\bar{\sigma} = \left(\frac{1}{T-t}\int_t^T \sigma^2(\zeta)d\zeta\right)^{1/2}.$$

The Greeks are then calculated as follows:

$$\Delta = \frac{\partial f}{\partial S} = e^{-\hat{q}(T-t)}N(d_1) + \frac{Se^{-\hat{q}(T-t)}N'(d_1) - Xe^{-\hat{r}(T-t)}N'(d_2)}{\bar{\sigma}S\sqrt{T-t}},$$

$$\Gamma = \frac{\partial^2 f}{\partial S^2} = \frac{Se^{-\hat{q}(T-t)}N'(d_1) + Xe^{-\hat{r}(T-t)}N'(d_2)}{\bar{\sigma}S^2\sqrt{T-t}} + \frac{Se^{-\hat{q}(T-t)}N'(d_1) - Xe^{-\hat{r}(T-t)}N'(d_2)}{\bar{\sigma}^2S^2(T-t)},$$

$$\Theta = \frac{\partial f}{\partial t} = rf + (q-r)S\Delta - \frac{\sigma^2S^2}{2}\Gamma,$$

$$\Lambda = \frac{\partial f}{\partial \sigma} = \left(\frac{Xd_1e^{-\hat{r}(T-t)}N'(d_2) - Sd_2e^{-\hat{q}(T-t)}N'(d_1)}{\bar{\sigma}^2}\right)\hat{\sigma},$$

$$\rho = \frac{\partial f}{\partial r} = X(T-t)e^{-\hat{r}(T-t)}N(d_2) + \frac{\left(Se^{-\hat{q}(T-t)}N'(d_1) - Xe^{-\hat{r}(T-t)}N'(d_2)\right)\sqrt{T-t}}{\bar{\sigma}}.$$

Note: that Θ is obtained from substitution of other Greeks in the Black–Scholes partial differential equation, rather than differentiation of f. The values of q, r and σ appearing in its definition are the instantaneous values, not the averages. Note also that both the first-order average $\hat{\sigma}$ and the second-order average $\bar{\sigma}$ appear in the expression for Λ. This results from the fact that Λ is the derivative of f with respect to σ, not $\hat{\sigma}$.

For a European put option the equivalent equations are:

$$f = Xe^{-\hat{r}(T-t)}N(-d_2) - Se^{-\hat{q}(T-t)}N(-d_1),$$

$$\Delta = \frac{\partial f}{\partial S} = -e^{-\hat{q}(T-t)}N(-d_1) + \frac{Se^{-\hat{q}(T-t)}N'(-d_1) - Xe^{-\hat{r}(T-t)}N'(-d_2)}{\bar{\sigma}S\sqrt{T-t}},$$

$$\Gamma = \frac{\partial^2 f}{\partial S^2} = \frac{Xe^{-\hat{r}(T-t)}N'(-d_2) + Se^{-\hat{q}(T-t)}N'(-d_1)}{\bar{\sigma}S^2\sqrt{T-t}} + \frac{Xe^{-\hat{r}(T-t)}N''(-d_2) - Se^{-\hat{q}(T-t)}N''(-d_1)}{\bar{\sigma}^2S^2(T-t)},$$

$$\Theta = \frac{\partial f}{\partial t} = rf + (q-r)S\Delta - \frac{\sigma^2S^2}{2}\Gamma,$$

$$\Lambda = \frac{\partial f}{\partial \sigma} = \left(\frac{Xd_1e^{-\hat{r}(T-t)}N'(-d_2) - Sd_2e^{-\hat{q}(T-t)}N'(-d_1)}{\bar{\sigma}^2}\right)\hat{\sigma},$$

$$\rho = \frac{\partial f}{\partial r} = -X(T-t)e^{-\hat{r}(T-t)}N(-d_2) + \frac{\left(Se^{-\hat{q}(T-t)}N'(-d_1) - Xe^{-\hat{r}(T-t)}N'(-d_2)\right)\sqrt{T-t}}{\hat{\sigma}}.$$

The analytic solution for an American call option with $q = 0$ is identical to that for a European call, since early exercise is never optimal in this case. For all other cases no analytic solution is known.

10 Example

This example solves the Black–Scholes equation for valuation of a 5-month American call option on a non-dividend-paying stock with an exercise price of \$50. The risk-free interest rate is 10% per annum, and the stock volatility is 40% per annum.

The option is valued at a range of times and stock prices.

10.1 Program Text

```
/* nag_pde_bs_1d_analytic (d03ndc) Example Program.
 *
 * NAGPRODCODE Version.
 *
 * Copyright 2016 Numerical Algorithms Group.
 *
 * Mark 26, 2016.
 */

#include <stdio.h>
#include <string.h>
#include <math.h>
#include <nag.h>
#include <nag_stdlib.h>
#include <nagd03.h>

#define F(I, J)      f[ns*((J) -1)+(I) -1]
#define THETA(I, J)  theta[ns*((J) -1)+(I) -1]
#define DELTA(I, J)  delta[ns*((J) -1)+(I) -1]
#define GAMMA(I, J)  gamma[ns*((J) -1)+(I) -1]
```

```
#define LAMBDA(I, J) lambda[ns*((J) -1)+(I) -1]
#define RHO(I, J)    rho[ns*((J) -1)+(I) -1]

int main(void)
{
  double ds, dt, tmat, x;
  Integer i, igreek, j, ns, nt, exit_status = 0;
  double *delta = 0, *f = 0, *gamma = 0, *lambda = 0, q[3], r[3],
         *rho = 0, *s = 0;
  double sigma[3], *t = 0, *theta = 0, smin, smax, tmin, tmax;
  Nag_Boolean gprnt[5] = { Nag_TRUE, Nag_TRUE, Nag_TRUE, Nag_TRUE, Nag_TRUE };
  Nag_Boolean tdpar[3];
  const char *gname[5] = { "Theta", "Delta", "Gamma", "Lambda", "Rho" };
  NagError fail;

  INIT_FAIL(fail);

  printf("nag_pde_bs_1d_analytic (d03ndc) Example Program Results\n\n");

  /* Skip heading in data file */
#ifdef _WIN32
  scanf_s("%*[^\n] ");
#else
  scanf("%*[^\n] ");
#endif

  /* Read problem parameters */
#ifdef _WIN32
  scanf_s("%lf", &x);
#else
  scanf("%lf", &x);
#endif
#ifdef _WIN32
  scanf_s("%lf", &tmat);
#else
  scanf("%lf", &tmat);
#endif
#ifdef _WIN32
  scanf_s("%lf", &r[0]);
#else
  scanf("%lf", &r[0]);
#endif
#ifdef _WIN32
  scanf_s("%lf", &q[0]);
#else
  scanf("%lf", &q[0]);
#endif
#ifdef _WIN32
  scanf_s("%lf", &sigma[0]);
#else
  scanf("%lf", &sigma[0]);
#endif
#ifdef _WIN32
  scanf_s("%" NAG_IFMT "%" NAG_IFMT "", &ns, &nt);
#else
  scanf("%" NAG_IFMT "%" NAG_IFMT "", &ns, &nt);
#endif
#ifdef _WIN32
  scanf_s("%lf%lf", &smin, &smax);
#else
  scanf("%lf%lf", &smin, &smax);
#endif
#ifdef _WIN32
  scanf_s("%lf%lf", &tmin, &tmax);
#else
  scanf("%lf%lf", &tmin, &tmax);
#endif

  /* Allocate memory */

  if (!(s = NAG_ALLOC(ns, double)) ||
```

```
    !(t = NAG_ALLOC(nt, double)) ||
    !(f = NAG_ALLOC(ns * nt, double)) ||
    !(theta = NAG_ALLOC(ns * nt, double)) ||
    !(delta = NAG_ALLOC(ns * nt, double)) ||
    !(gamma = NAG_ALLOC(ns * nt, double)) ||
    !(lambda = NAG_ALLOC(ns * nt, double)) ||
    !(rho = NAG_ALLOC(ns * nt, double)))
{
  printf("Allocation failure\n");
  exit_status = 1;
  goto END;
}

/* Set up input parameters for nag_pde_bs_1d (d03ncc) */

s[0] = smin;
s[ns - 1] = smax;
t[0] = tmin;
t[nt - 1] = tmax;
tdpar[0] = Nag_FALSE;
tdpar[1] = Nag_FALSE;
tdpar[2] = Nag_FALSE;

ds = (s[ns - 1] - s[0]) / (ns - 1.0);
dt = (t[nt - 1] - t[0]) / (nt - 1.0);

/* Loop over times */

for (j = 1; j <= nt; j++) {
  t[j - 1] = t[0] + (j - 1) * dt;

  /* Loop over stock prices */

  for (i = 1; i <= ns; i++) {
    s[i - 1] = s[0] + (i - 1) * ds;

    /* Evaluate analytic solution of Black-Scholes equation */

    /* nag_pde_bs_1d_analytic (d03ndc).
     * Analytic solution of the Black-Scholes equations
     */
    nag_pde_bs_1d_analytic(Nag_AmericanCall, x, s[i - 1], t[j - 1], tmat,
                           tdpar, r, q, sigma, &F(i, j), &THETA(i, j),
                           &DELTA(i, j), &GAMMA(i, j), &LAMBDA(i, j),
                           &RHO(i, j), &fail);

    if (fail.code != NE_NOERROR) {
      printf("Error from nag_pde_bs_1d_analytic (d03ndc).\n%s\n",
              fail.message);
      exit_status = 1;
      goto END;
    }
  }
}

/* Output option values */

printf("\n");
printf("Option Values\n");
printf("-------------\n");
printf("%14s  |  %s\n", "Stock Price", "Time to Maturity (months)");
printf("%14s  | ", "");
for (i = 0; i < nt; i++)
  printf(" %13.4e", 12.0 * (tmat - t[i]));
printf("\n");
for (i = 0; i < 74; i++)
  printf("-");
printf("\n");
for (i = 1; i <= ns; i++) {
  printf(" %13.4e  | ", s[i - 1]);
  for (j = 1; j <= nt; j++)
```

```
      printf(" %13.4e", F(i, j));
    printf("\n");
  }

  for (igreek = 0; igreek < 5; igreek++) {
    if (!gprnt[igreek])
      continue;

    printf("\n");
    printf("%s\n", gname[igreek]);
    for (i = 0; i < (Integer) strlen(gname[igreek]); i++)
      printf("-");
    printf("\n");
    printf("%14s  |  %s\n", "Stock Price", "Time to Maturity (months)");
    printf("%14s  | ", "");
    for (i = 0; i < nt; i++)
      printf(" %13.4e", 12.0 * (tmat - t[i]));
    printf("\n");
    for (i = 0; i < 74; i++)
      printf("-");
    printf("\n");

    for (i = 1; i <= ns; i++) {
      printf(" %13.4e  | ", s[i - 1]);
      switch (igreek) {
      case 0:
        for (j = 1; j <= nt; j++)
          printf(" %13.4e", THETA(i, j));
        break;
      case 1:
        for (j = 1; j <= nt; j++)
          printf(" %13.4e", DELTA(i, j));
        break;
      case 2:
        for (j = 1; j <= nt; j++)
          printf(" %13.4e", GAMMA(i, j));
        break;
      case 3:
        for (j = 1; j <= nt; j++)
          printf(" %13.4e", LAMBDA(i, j));
        break;
      case 4:
        for (j = 1; j <= nt; j++)
          printf(" %13.4e", RHO(i, j));
        break;
      default:
        break;
      }
      printf("\n");
    }
  }
END:
  NAG_FREE(s);
  NAG_FREE(t);
  NAG_FREE(f);
  NAG_FREE(theta);
  NAG_FREE(delta);
  NAG_FREE(gamma);
  NAG_FREE(lambda);
  NAG_FREE(rho);

  return exit_status;
}
```

10.2 Program Data

```
nag_pde_bs_1d_analytic (d03ndc) Example Program Data
  50.
  0.4166667
  0.1
  0.0
  0.4
  21  4
  0.0  100.
  0.0  0.125
```

10.3 Program Results

```
nag_pde_bs_1d_analytic (d03ndc) Example Program Results


Option Values
-------------
   Stock Price  |  Time to Maturity (months)
                |     5.0000e+00    4.5000e+00    4.0000e+00    3.5000e+00
-------------------------------------------------------------------------
    0.0000e+00  |     0.0000e+00    0.0000e+00    0.0000e+00    0.0000e+00
    5.0000e+00  |     4.4491e-19    4.5989e-21    1.5461e-23    1.0478e-26
    1.0000e+01  |     5.5566e-10    5.5129e-11    3.1298e-12    8.0281e-14
    1.5000e+01  |     4.7337e-06    1.2187e-06    2.2774e-07    2.7003e-08
    2.0000e+01  |     7.2236e-04    3.1054e-04    1.1005e-04    2.9678e-05
    2.5000e+01  |     1.6557e-02    9.6610e-03    5.0099e-03    2.2012e-03
    3.0000e+01  |     1.3307e-01    9.4037e-02    6.1869e-02    3.6848e-02
    3.5000e+01  |     5.6631e-01    4.5257e-01    3.4667e-01    2.5053e-01
    4.0000e+01  |     1.6004e+00    1.3850e+00    1.1699e+00    9.5640e-01
    4.5000e+01  |     3.4384e+00    3.1328e+00    2.8168e+00    2.4891e+00
    5.0000e+01  |     6.1165e+00    5.7600e+00    5.3874e+00    4.9960e+00
    5.5000e+01  |     9.5300e+00    9.1645e+00    8.7846e+00    8.3882e+00
    6.0000e+01  |     1.3509e+01    1.3163e+01    1.2808e+01    1.2445e+01
    6.5000e+01  |     1.7883e+01    1.7568e+01    1.7251e+01    1.6932e+01
    7.0000e+01  |     2.2513e+01    2.2230e+01    2.1949e+01    2.1671e+01
    7.5000e+01  |     2.7301e+01    2.7045e+01    2.6792e+01    2.6544e+01
    8.0000e+01  |     3.2182e+01    3.1946e+01    3.1713e+01    3.1485e+01
    8.5000e+01  |     3.7117e+01    3.6894e+01    3.6674e+01    3.6458e+01
    9.0000e+01  |     4.2081e+01    4.1868e+01    4.1656e+01    4.1446e+01
    9.5000e+01  |     4.7062e+01    4.6854e+01    4.6647e+01    4.6441e+01
    1.0000e+02  |     5.2052e+01    5.1847e+01    5.1643e+01    5.1439e+01

Theta
-----
   Stock Price  |  Time to Maturity (months)
                |     5.0000e+00    4.5000e+00    4.0000e+00    3.5000e+00
-------------------------------------------------------------------------
    0.0000e+00  |     0.0000e+00    0.0000e+00    0.0000e+00    0.0000e+00
    5.0000e+00  |    -4.4017e-17   -5.5977e-19   -2.3735e-21   -2.0936e-24
    1.0000e+01  |    -2.7827e-08   -3.3857e-09   -2.4163e-10   -8.0398e-12
    1.5000e+01  |    -1.3953e-04   -4.3864e-05   -1.0258e-05   -1.5706e-06
    2.0000e+01  |    -1.3287e-02   -6.9342e-03   -3.0567e-03   -1.0576e-03
    2.5000e+01  |    -1.9512e-01   -1.3714e-01   -8.7730e-02   -4.9018e-02
    3.0000e+01  |    -1.0161e+00   -8.5596e-01   -6.8695e-01   -5.1395e-01
    3.5000e+01  |    -2.8112e+00   -2.6426e+00   -2.4328e+00   -2.1723e+00
    4.0000e+01  |    -5.1662e+00   -5.1709e+00   -5.1500e+00   -5.0892e+00
    4.5000e+01  |    -7.2196e+00   -7.4540e+00   -7.7180e+00   -8.0183e+00
    5.0000e+01  |    -8.3848e+00   -8.7388e+00   -9.1543e+00   -9.6525e+00
    5.5000e+01  |    -8.6152e+00   -8.9372e+00   -9.3056e+00   -9.7329e+00
    6.0000e+01  |    -8.2058e+00   -8.4077e+00   -8.6186e+00   -8.8343e+00
    6.5000e+01  |    -7.5116e+00   -7.5845e+00   -7.6368e+00   -7.6553e+00
    7.0000e+01  |    -6.7905e+00   -6.7711e+00   -6.7202e+00   -6.6262e+00
    7.5000e+01  |    -6.1758e+00   -6.1099e+00   -6.0160e+00   -5.8893e+00
    8.0000e+01  |    -5.7084e+00   -5.6310e+00   -5.5359e+00   -5.4234e+00
    8.5000e+01  |    -5.3786e+00   -5.3103e+00   -5.2340e+00   -5.1533e+00
    9.0000e+01  |    -5.1582e+00   -5.1071e+00   -5.0551e+00   -5.0062e+00
    9.5000e+01  |    -5.0165e+00   -4.9835e+00   -4.9536e+00   -4.9298e+00
    1.0000e+02  |    -4.9281e+00   -4.9107e+00   -4.8979e+00   -4.8916e+00
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Delta

Stock Price	Time to Maturity (months)			
	5.0000e+00	4.5000e+00	4.0000e+00	3.5000e+00
0.0000e+00	0.0000e+00	0.0000e+00	0.0000e+00	0.0000e+00
5.0000e+00	3.1381e-18	3.5969e-20	1.3576e-22	1.0494e-25
1.0000e+01	1.4005e-09	1.5376e-10	9.7805e-12	2.8553e-13
1.5000e+01	6.1418e-06	1.7452e-06	3.6436e-07	4.9030e-08
2.0000e+01	5.6040e-04	2.6494e-04	1.0451e-04	3.1863e-05
2.5000e+01	8.3312e-03	5.3217e-03	3.0570e-03	1.5104e-03
3.0000e+01	4.5711e-02	3.5158e-02	2.5461e-02	1.6934e-02
3.5000e+01	1.3765e-01	1.1889e-01	9.9459e-02	7.9557e-02
4.0000e+01	2.8307e-01	2.6258e-01	2.3996e-01	2.1479e-01
4.5000e+01	4.5320e-01	4.3858e-01	4.2214e-01	4.0335e-01
5.0000e+01	6.1427e-01	6.0856e-01	6.0249e-01	5.9601e-01
5.5000e+01	7.4525e-01	7.4687e-01	7.4937e-01	7.5308e-01
6.0000e+01	8.4052e-01	8.4611e-01	8.5298e-01	8.6148e-01
6.5000e+01	9.0433e-01	9.1096e-01	9.1862e-01	9.2752e-01
7.0000e+01	9.4449e-01	9.5045e-01	9.5699e-01	9.6412e-01
7.5000e+01	9.6862e-01	9.7325e-01	9.7808e-01	9.8300e-01
8.0000e+01	9.8260e-01	9.8589e-01	9.8913e-01	9.9221e-01
8.5000e+01	9.9050e-01	9.9269e-01	9.9473e-01	9.9653e-01
9.0000e+01	9.9487e-01	9.9627e-01	9.9748e-01	9.9848e-01
9.5000e+01	9.9725e-01	9.9811e-01	9.9881e-01	9.9935e-01
1.0000e+02	9.9854e-01	9.9905e-01	9.9945e-01	9.9972e-01

Gamma

Stock Price	Time to Maturity (months)			
	5.0000e+00	4.5000e+00	4.0000e+00	3.5000e+00
0.0000e+00	0.0000e+00	0.0000e+00	0.0000e+00	0.0000e+00
5.0000e+00	2.1246e-17	2.7112e-19	1.1536e-21	1.0211e-24
1.0000e+01	3.3102e-09	4.0468e-10	2.9020e-11	9.7029e-13
1.5000e+01	7.2660e-06	2.2982e-06	5.4080e-07	8.3319e-08
2.0000e+01	3.8245e-04	2.0111e-04	8.9333e-05	3.1153e-05
2.5000e+01	3.5190e-03	2.4960e-03	1.6118e-03	9.0924e-04
3.0000e+01	1.2392e-02	1.0554e-02	8.5660e-03	6.4838e-03
3.5000e+01	2.4348e-02	2.3181e-02	2.1626e-02	1.9580e-02
4.0000e+01	3.2765e-02	3.3274e-02	3.3650e-02	3.3795e-02
4.5000e+01	3.4099e-02	3.5763e-02	3.7655e-02	3.9828e-02
5.0000e+01	2.9625e-02	3.1360e-02	3.3403e-02	3.5860e-02
5.5000e+01	2.2600e-02	2.3743e-02	2.5052e-02	2.6569e-02
6.0000e+01	1.5672e-02	1.6137e-02	1.6603e-02	1.7048e-02
6.5000e+01	1.0123e-02	1.0119e-02	1.0032e-02	9.8216e-03
7.0000e+01	6.1999e-03	5.9720e-03	5.6534e-03	5.2154e-03
7.5000e+01	3.6474e-03	3.3666e-03	3.0215e-03	2.6027e-03
8.0000e+01	2.0815e-03	1.8329e-03	1.5510e-03	1.2387e-03
8.5000e+01	1.1610e-03	9.7196e-04	7.7211e-04	5.6851e-04
9.0000e+01	6.3660e-04	5.0529e-04	3.7553e-04	2.5382e-04
9.5000e+01	3.4468e-04	2.5884e-04	1.7950e-04	1.1099e-04
1.0000e+02	1.8494e-04	1.3118e-04	8.4708e-05	4.7786e-05

Lambda

Stock Price	Time to Maturity (months)			
	5.0000e+00	4.5000e+00	4.0000e+00	3.5000e+00
0.0000e+00	0.0000e+00	0.0000e+00	0.0000e+00	0.0000e+00
5.0000e+00	8.8525e-17	1.0167e-18	3.8453e-21	2.9781e-24
1.0000e+01	5.5171e-08	6.0702e-09	3.8694e-10	1.1320e-11
1.5000e+01	2.7247e-04	7.7565e-05	1.6224e-05	2.1871e-06
2.0000e+01	2.5496e-02	1.2066e-02	4.7644e-03	1.4538e-03
2.5000e+01	3.6656e-01	2.3400e-01	1.3431e-01	6.6299e-02
3.0000e+01	1.8588e+00	1.4248e+00	1.0279e+00	6.8080e-01
3.5000e+01	4.9710e+00	4.2595e+00	3.5323e+00	2.7983e+00
4.0000e+01	8.7374e+00	7.9857e+00	7.1787e+00	6.3084e+00
4.5000e+01	1.1508e+01	1.0863e+01	1.0167e+01	9.4094e+00
5.0000e+01	1.2344e+01	1.1760e+01	1.1134e+01	1.0459e+01

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    5.5000e+01  |     1.1394e+01    1.0773e+01    1.0104e+01    9.3768e+00
    6.0000e+01  |     9.4033e+00    8.7137e+00    7.9693e+00    7.1602e+00
    6.5000e+01  |     7.1285e+00    6.4127e+00    5.6514e+00    4.8412e+00
    7.0000e+01  |     5.0632e+00    4.3894e+00    3.6936e+00    2.9815e+00
    7.5000e+01  |     3.4194e+00    2.8406e+00    2.2661e+00    1.7080e+00
    8.0000e+01  |     2.2203e+00    1.7596e+00    1.3235e+00    9.2488e-01
    8.5000e+01  |     1.3981e+00    1.0534e+00    7.4380e-01    4.7920e-01
    9.0000e+01  |     8.5941e-01    6.1393e-01    4.0558e-01    2.3986e-01
    9.5000e+01  |     5.1846e-01    3.5040e-01    2.1600e-01    1.1686e-01
    1.0000e+02  |     3.0824e-01    1.9677e-01    1.1294e-01    5.5750e-02

Rho
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   Stock Price  |  Time to Maturity (months)
                |     5.0000e+00    4.5000e+00    4.0000e+00    3.5000e+00
-------------------------------------------------------------------------
    0.0000e+00  |     0.0000e+00    0.0000e+00    0.0000e+00    0.0000e+00
    5.0000e+00  |     6.3524e-18    6.5717e-20    2.2112e-22    1.4997e-25
    1.0000e+01  |     5.6040e-09    5.5594e-10    3.1558e-11    8.0937e-13
    1.5000e+01  |     3.6414e-05    9.3595e-06    1.7459e-06    2.0663e-07
    2.0000e+01  |     4.3690e-03    1.8706e-03    6.6008e-04    1.7721e-04
    2.5000e+01  |     7.9884e-02    4.6268e-02    2.3805e-02    1.0371e-02
    3.0000e+01  |     5.1594e-01    3.6026e-01    2.3399e-01    1.3743e-01
    3.5000e+01  |     1.7715e+00    1.3907e+00    1.0448e+00    7.3907e-01
    4.0000e+01  |     4.0509e+00    3.4193e+00    2.8095e+00    2.2269e+00
    4.5000e+01  |     7.0648e+00    6.2263e+00    5.3932e+00    4.5679e+00
    5.0000e+01  |     1.0249e+01    9.2505e+00    8.2458e+00    7.2346e+00
    5.5000e+01  |     1.3108e+01    1.1967e+01    1.0810e+01    9.6342e+00
    6.0000e+01  |     1.5384e+01    1.4101e+01    1.2790e+01    1.1446e+01
    6.5000e+01  |     1.7041e+01    1.5617e+01    1.4153e+01    1.2646e+01
    7.0000e+01  |     1.8167e+01    1.6613e+01    1.5013e+01    1.3363e+01
    7.5000e+01  |     1.8894e+01    1.7231e+01    1.5521e+01    1.3761e+01
    8.0000e+01  |     1.9344e+01    1.7597e+01    1.5806e+01    1.3969e+01
    8.5000e+01  |     1.9615e+01    1.7807e+01    1.5959e+01    1.4072e+01
    9.0000e+01  |     1.9774e+01    1.7924e+01    1.6039e+01    1.4122e+01
    9.5000e+01  |     1.9865e+01    1.7987e+01    1.6080e+01    1.4145e+01
    1.0000e+02  |     1.9917e+01    1.8022e+01    1.6101e+01    1.4156e+01
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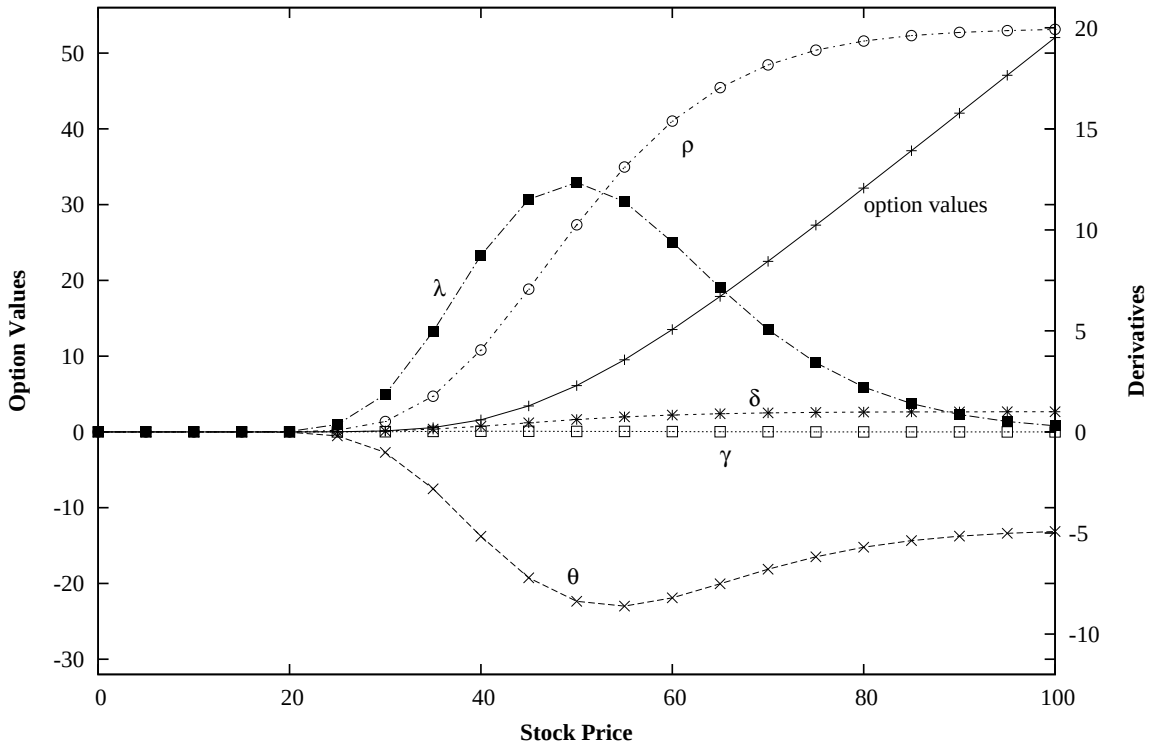
Example Program 1
Option Values and Derivatives at 5 Months to Maturity



Example Program 2
Option Values and Derivatives at 3.5 Months to Maturity