

# NAG Library Function Document

## nag\_ode\_ivp\_adams\_gen (d02cjc)

### 1 Purpose

nag\_ode\_ivp\_adams\_gen (d02cjc) integrates a system of first order ordinary differential equations over a range with suitable initial conditions, using a variable-order, variable-step Adams' method until a user-specified function, if supplied, of the solution is zero, and returns the solution at specified points, if desired.

### 2 Specification

```
#include <nag.h>
#include <nagd02.h>

void nag_ode_ivp_adams_gen (Integer neq,
    void (*fcn)(Integer neq, double x, const double y[], double f[],
        Nag_User *comm),
    double *x, double y[], double xend, double tol, Nag_ErrorControl err_c,
    void (*output)(Integer neq, double *xsol, const double y[],
        Nag_User *comm),
    double (*g)(Integer neq, double x, const double y[], Nag_User *comm),
    Nag_User *comm, NagError *fail)
```

### 3 Description

nag\_ode\_ivp\_adams\_gen (d02cjc) advances the solution of a system of ordinary differential equations

$$y'_i = f_i(x, y_1, y_2, \dots, y_{\text{neq}}), \quad i = 1, 2, \dots, \text{neq},$$

from  $x = \mathbf{x}$  to  $x = \mathbf{xend}$  using a variable-order, variable-step Adams' method. The system is defined by **fcn**, which evaluates  $f_i$  in terms of  $x$  and  $y_1, y_2, \dots, y_{\text{neq}}$ . The initial values of  $y_1, y_2, \dots, y_{\text{neq}}$  must be given at  $x = \mathbf{x}$ .

The solution is returned via **output** at specified points, if desired: this solution is obtained by  $C^1$  interpolation on solution values produced by the method. As the integration proceeds a check can be made on the user-specified function  $g(x, y)$  to determine an interval where it changes sign. The position of this sign change is then determined accurately. It is assumed that  $g(x, y)$  is a continuous function of the variables, so that a solution of  $g(x, y) = 0.0$  can be determined by searching for a change in sign in  $g(x, y)$ . The accuracy of the integration, the interpolation and, indirectly, of the determination of the position where  $g(x, y) = 0.0$ , is controlled by the arguments **tol** and **err\_c**.

For a description of Adams' methods and their practical implementation see Hall and Watt (1976).

### 4 References

Hall G and Watt J M (ed.) (1976) *Modern Numerical Methods for Ordinary Differential Equations* Clarendon Press, Oxford

### 5 Arguments

- 1: **neq** – Integer *Input*  
*On entry:* the number of differential equations.  
*Constraint:* **neq**  $\geq 1$ .

- 2: **fcn** – function, supplied by the user *External Function*

**fcn** must evaluate the first derivatives  $y'_i$  (i.e., the functions  $f_i$ ) for given values of their arguments  $x, y_1, y_2, \dots, y_{\text{neq}}$ .

The specification of **fcn** is:

```
void fcn (Integer neq, double x, const double y[], double f[],
         Nag_User *comm)
```

1: **neq** – Integer *Input*

*On entry:* the number of differential equations.

2: **x** – double *Input*

*On entry:* the value of the independent variable  $x$ .

3: **y[neq]** – const double *Input*

*On entry:* **y**[ $i - 1$ ] holds the value of the variable  $y_i$ , for  $i = 1, 2, \dots, \text{neq}$ .

4: **f[neq]** – double *Output*

*On exit:* **f**[ $i - 1$ ] must contain the value of  $f_i$ , for  $i = 1, 2, \dots, \text{neq}$ .

5: **comm** – Nag\_User \*

Pointer to a structure of type Nag\_User with the following member:

**p** – Pointer

*On entry/exit:* the pointer **comm**→**p** should be cast to the required type, e.g.,  
`struct user *s = (struct user *)comm → p`, to obtain the original  
object's address with appropriate type. (See the argument **comm** below.)

- 3: **x** – double \* *Input/Output*

*On entry:* the initial value of the independent variable  $x$ .

*Constraint:* **x**  $\neq$  **xend**.

*On exit:* if  $g$  is supplied, **x** contains the point where  $g(x, y) = 0.0$ , unless  $g(x, y) \neq 0.0$  anywhere on the range **x** to **xend**, in which case, **x** will contain **xend**. If  $g$  is not supplied **x** contains **xend**, unless an error has occurred, when it contains the value of  $x$  at the error.

- 4: **y[neq]** – double *Input/Output*

*On entry:* the initial values of the solution  $y_1, y_2, \dots, y_{\text{neq}}$  at  $x = \mathbf{x}$ .

*On exit:* the computed values of the solution at the final point  $x = \mathbf{x}$ .

- 5: **xend** – double *Input*

*On entry:* the final value of the independent variable.

**xend** < **x**

Integration proceeds in the negative direction.

*Constraint:* **xend**  $\neq$  **x**.

- 6: **tol** – double *Input*

*On entry:* a positive tolerance for controlling the error in the integration. Hence **tol** affects the determination of the position where  $g(x, y) = 0.0$ , if  $g$  is supplied.

`nag_ode_ivp_adams_gen` (d02cjc) has been designed so that, for most problems, a reduction in **tol** leads to an approximately proportional reduction in the error in the solution. However, the actual relation between **tol** and the accuracy achieved cannot be guaranteed. You are strongly recommended to call `nag_ode_ivp_adams_gen` (d02cjc) with more than one value for **tol** and to compare the results obtained to estimate their accuracy. In the absence of any prior knowledge, you might compare the results obtained by calling `nag_ode_ivp_adams_gen` (d02cjc) with **tol** =  $10.0^{-p}$  and **tol** =  $10.0^{-p-1}$  where  $p$  correct decimal digits are required in the solution.

*Constraint:* **tol** > 0.0.

7: **err\_c** – Nag\_ErrorControl

*Input*

*On entry:* the type of error control. At each step in the numerical solution an estimate of the local error, *est*, is made. For the current step to be accepted the following condition must be satisfied:

$$est = \sqrt{\sum_{i=1}^{neq} (e_i / (\tau_r \times |y_i| + \tau_a))^2} \leq 1.0$$

where  $\tau_r$  and  $\tau_a$  are defined by

<b>err_c</b>	$\tau_r$	$\tau_a$
Nag_Relative	<b>tol</b>	$\epsilon$
Nag_Absolute	0.0	<b>tol</b>
Nag_Mixed	<b>tol</b>	<b>tol</b>

where  $\epsilon$  is a small machine-dependent number and  $e_i$  is an estimate of the local error at  $y_i$ , computed internally. If the appropriate condition is not satisfied, the step size is reduced and the solution is recomputed on the current step. If you wish to measure the error in the computed solution in terms of the number of correct decimal places, then **err\_c** should be set to Nag\_Absolute on entry, whereas if the error requirement is in terms of the number of correct significant digits, then **err\_c** should be set to Nag\_Relative. If you prefer a mixed error test, then **err\_c** should be set to Nag\_Mixed. The recommended value for **err\_c** is Nag\_Mixed and this should be chosen unless there are good reasons for a different choice.

*Constraint:* **err\_c** = Nag\_Relative, Nag\_Absolute or Nag\_Mixed.

8: **output** – function, supplied by the user

*External Function*

**output** permits access to intermediate values of the computed solution (for example to print or plot them), at successive user-specified points. It is initially called by `nag_ode_ivp_adams_gen` (d02cjc) with **xsol** = **x** (the initial value of  $x$ ). You must reset **xsol** to the next point (between the current **xsol** and **xend**) where **output** is to be called, and so on at each call to **output**. If, after a call to **output**, the reset point **xsol** is beyond **xend**, `nag_ode_ivp_adams_gen` (d02cjc) will integrate to **xend** with no further calls to **output**; if a call to **output** is required at the point **xsol** = **xend**, then **xsol** must be given precisely the value **xend**.

The specification of **output** is:

```
void output (Integer neq, double *xsol, const double y[],
            Nag_User *comm)
```

1: **neq** – Integer

*Input*

*On entry:* the number of differential equations.

2: **xsol** – double \*

*Input/Output*

*On entry:* the value of the independent variable  $x$ .

*On exit:* you must set **xsol** to the next value of  $x$  at which **output** is to be called.

3:	<b>y[neq]</b> – const double	<i>Input</i>
	<i>On entry:</i> the computed solution at the point <b>xsol</b> .	
4:	<b>comm</b> – Nag_User *	
	Pointer to a structure of type Nag_User with the following member:	
	<b>p</b> – Pointer	
	<i>On entry/exit:</i> the pointer <b>comm</b> → <b>p</b> should be cast to the required type, e.g., <pre>struct user *s = (struct user *)comm → p,</pre> to obtain the original object's address with appropriate type. (See the argument <b>comm</b> below.)	

If you do not wish to access intermediate output, the actual argument **output** must be the NAG defined null function pointer `NULLFN`.

- 9: **g** – function, supplied by the user *External Function*
- g** must evaluate  $g(x,y)$  for specified values  $x,y$ . It specifies the function  $g$  for which the first position  $x$  where  $g(x,y) = 0$  is to be found.

The specification of <b>g</b> is:		
<code>double g (Integer neq, double x, const double y[], Nag_User *comm)</code>		
1:	<b>neq</b> – Integer	<i>Input</i>
	<i>On entry:</i> the number of differential equations.	
2:	<b>x</b> – double	<i>Input</i>
	<i>On entry:</i> the value of the independent variable $x$ .	
3:	<b>y[neq]</b> – const double	<i>Input</i>
	<i>On entry:</i> <b>y</b> [ $i - 1$ ] holds the value of the variable $y_i$ , for $i = 1, 2, \dots, \mathbf{neq}$ .	
4:	<b>comm</b> – Nag_User *	
	Pointer to a structure of type Nag_User with the following member:	
	<b>p</b> – Pointer	
	<i>On entry/exit:</i> the pointer <b>comm</b> → <b>p</b> should be cast to the required type, e.g., <pre>struct user *s = (struct user *)comm → p,</pre> to obtain the original object's address with appropriate type. (See the argument <b>comm</b> below.)	

If you do not require the root finding option, the actual argument **g** must be the NAG defined null double function pointer `NULLDFN`.

- 10: **comm** – Nag\_User \*
- Pointer to a structure of type Nag\_User with the following member:
- p** – Pointer
- On entry/exit:* the pointer **comm**→**p**, of type Pointer, allows you to communicate information to and from **fcn**, **output** and **g**. An object of the required type should be declared, e.g., a structure, and its address assigned to the pointer **comm**→**p** by means of a cast to Pointer in the calling program. E.g. `comm.p = (Pointer)&s`. The type pointer will be `void *` with a C compiler that defines `void *` and `char *` otherwise.

11: **fail** – NagError \*

*Input/Output*

The NAG error argument (see Section 2.7 in How to Use the NAG Library and its Documentation).

## 6 Error Indicators and Warnings

### NE\_2\_REAL\_ARG\_EQ

On entry,  $x = \langle value \rangle$  while  $xend = \langle value \rangle$ . These arguments must satisfy  $x \neq xend$ .

### NE\_ALLOC\_FAIL

Dynamic memory allocation failed.

### NE\_BAD\_PARAM

On entry, argument **err\_c** had an illegal value.

### NE\_INT\_ARG\_LT

On entry, **neq** =  $\langle value \rangle$ .

Constraint: **neq**  $\geq 1$ .

### NE\_INTERNAL\_ERROR

An internal error has occurred in this function. Check the function call and any array sizes. If the call is correct then please contact NAG for assistance.

### NE\_NO\_SIGN\_CHANGE

No change in sign of the function  $g(x, y)$  was detected in the integration range.

### NE\_REAL\_ARG\_LE

On entry, **tol** must not be less than or equal to 0.0: **tol** =  $\langle value \rangle$ .

### NE\_TOL\_PROGRESS

The value of **tol**,  $\langle value \rangle$ , is too small for the function to make any further progress across the integration range. Current value of  $x = \langle value \rangle$ .

### NE\_TOL\_TOO\_SMALL

The value of **tol**,  $\langle value \rangle$ , is too small for the function to take an initial step.

### NE\_XSOL\_INCONSIST

On call  $\langle value \rangle$  to the supplied print function **xsol** was set to a value behind the previous value of **xsol** in the direction of integration.

Previous **xsol** =  $\langle value \rangle$ , **xend** =  $\langle value \rangle$ , new **xsol** =  $\langle value \rangle$ .

### NE\_XSOL\_NOT\_RESET

On call  $\langle value \rangle$  to the supplied print function **xsol** was not reset.

### NE\_XSOL\_SET\_WRONG

**xsol** was set to a value behind  $x$  in the direction of integration by the first call to the supplied print function.

The integration range is  $[\langle value1 \rangle, \langle value2 \rangle]$ , **xsol** =  $\langle value \rangle$ .

## 7 Accuracy

The accuracy of the computation of the solution vector **y** may be controlled by varying the local error tolerance **tol**. In general, a decrease in local error tolerance should lead to an increase in accuracy. You are advised to choose **err\_c** = Nag\_Mixed unless you have a good reason for a different choice.

If the problem is a root-finding one, then the accuracy of the root determined will depend on the properties of  $g(x, y)$ . You should try to code **g** without introducing any unnecessary cancellation errors.

## 8 Parallelism and Performance

nag\_ode\_ivp\_adams\_gen (d02cjc) is not threaded in any implementation.

## 9 Further Comments

If more than one root is required then nag\_ode\_ivp\_adams\_roots (d02qfc) should be used.

If the function fails with error exit of **fail.code** = NE\_TOL\_TOO\_SMALL, then it can be called again with a larger value of **tol** if this has not already been tried. If the accuracy requested is really needed and cannot be obtained with this function, the system may be very stiff (see below) or so badly scaled that it cannot be solved to the required accuracy.

If the function fails with error exit of **fail.code** = NE\_TOL\_PROGRESS, it is probable that it has been called with a value of **tol** which is so small that a solution cannot be obtained on the range **x** to **xend**. This can happen for well-behaved systems and very small values of **tol**. You should, however, consider whether there is a more fundamental difficulty. For example:

- (a) in the region of a singularity (infinite value) of the solution, the function will usually stop with error exit of **fail.code** = NE\_TOL\_PROGRESS, unless overflow occurs first. Numerical integration cannot be continued through a singularity, and analytic treatment should be considered;
- (b) for 'stiff' equations where the solution contains rapidly decaying components, the function will use very small steps in  $x$  (internally to nag\_ode\_ivp\_adams\_gen (d02cjc)) to preserve stability. This will exhibit itself by making the computing time excessively long, or occasionally by an exit with **fail.code** = NE\_TOL\_PROGRESS. Adams' methods are not efficient in such cases.

## 10 Example

We illustrate the solution of four different problems. In each case the differential system (for a projectile) is

$$\begin{aligned} y' &= \tan \phi \\ v' &= \frac{-0.032 \tan \phi}{v} - \frac{0.02v}{\cos \phi} \\ \phi' &= \frac{-0.032}{v^2} \end{aligned}$$

over an interval **x** = 0.0 to **xend** = 10.0 starting with values  $y = 0.5$ ,  $v = 0.5$  and  $\phi = \pi/5$ . We solve each of the following problems with local error tolerances  $1.0\text{e-}4$  and  $1.0\text{e-}5$ .

- (i) To integrate to  $x = 10.0$  producing output at intervals of 2.0 until a point is encountered where  $y = 0.0$ .
- (ii) As (i) but with no intermediate output.
- (iii) As (i) but with no termination on a root-finding condition.
- (iv) As (i) but with no intermediate output and no root-finding termination condition.

## 10.1 Program Text

```

/* nag_ode_ivp_adams_gen (d02cjc) Example Program.
 *
 * NAGPRODCODE Version.
 *
 * Copyright 2016 Numerical Algorithms Group.
 *
 * Mark 26, 2016.
 *
 */

#include <nag.h>
#include <math.h>
#include <stdio.h>
#include <nag_stdlib.h>
#include <nagd02.h>
#include <nagx01.h>

#ifdef __cplusplus
extern "C"
{
#endif
    static void NAG_CALL out(Integer neq, double *xsol, const double y[],
                             Nag_User *comm);
    static void NAG_CALL fcn(Integer neq, double x, const double y[],
                             double f[], Nag_User *comm);
    static double NAG_CALL g(Integer neq, double x, const double y[],
                             Nag_User *comm);
#ifdef __cplusplus
}
#endif

struct user
{
    double xend, h;
    Integer k;
    Integer *use_comm;
};

int main(void)
{
    static Integer use_comm[2] = { 1, 1 };
    Integer exit_status = 0, i, j, neq;
    Nag_User comm;
    double pi, tol, x, y[3];
    struct user s;
    NagError fail;

    INIT_FAIL(fail);

    printf("nag_ode_ivp_adams_gen (d02cjc) Example Program Results\n");

    /* For communication with user-supplied functions
     * assign address of user defined structure
     * to Nag pointer.
     */
    s.use_comm = use_comm;
    comm.p = (Pointer) &s;

    neq = 3;
    s.xend = 10.0;
    /* nag_pi (x01aac).
     * pi
     */
    pi = nag_pi;
    printf("\nCase 1: intermediate output, root-finding\n");
    for (j = 4; j <= 5; ++j) {
        tol = pow(10.0, (double) (-j));
        printf("\n  Calculation with tol = %10.1e\n", tol);
        x = 0.0;
    }
}

```

```

y[0] = 0.5;
y[1] = 0.5;
y[2] = pi / 5.0;
s.k = 4;
s.h = (s.xend - x) / (double) (s.k + 1);
printf("\n      X          Y(1)          Y(2)          Y(3)\n");

/* nag_ode_ivp_adams_gen (d02cjc).
 * Ordinary differential equation solver using a
 * variable-order variable-step Adams method (Black Box)
 */
nag_ode_ivp_adams_gen(neq, fcn, &x, y, s.xend, tol, Nag_Mixed, out, g,
                      &comm, &fail);
if (fail.code != NE_NOERROR) {
    printf("Error from nag_ode_ivp_adams_gen (d02cjc).\n%s\n",
          fail.message);
    exit_status = 1;
    goto END;
}

printf("\n Root of Y(1) = 0.0 at %7.3f\n", x);
printf("\n Solution is");
for (i = 0; i < 3; ++i)
    printf("%10.5f", y[i]);
printf("\n");
}
printf("\n\nCase 2: no intermediate output, root-finding\n");
for (j = 4; j <= 5; ++j) {
    tol = pow(10.0, (double) (-j));
    printf("\n Calculation with tol = %10.1e\n", tol);
    x = 0.0;
    y[0] = 0.5;
    y[1] = 0.5;
    y[2] = pi / 5.0;

    /* nag_ode_ivp_adams_gen (d02cjc), see above. */
    nag_ode_ivp_adams_gen(neq, fcn, &x, y, s.xend, tol, Nag_Mixed, NULLFN, g,
                          &comm, &fail);
    if (fail.code != NE_NOERROR) {
        printf("Error from nag_ode_ivp_adams_gen (d02cjc).\n%s\n",
              fail.message);
        exit_status = 1;
        goto END;
    }
    printf("\n Root of Y(1) = 0.0 at %7.3f\n", x);
    printf("\n Solution is");
    for (i = 0; i < 3; ++i)
        printf("%10.5f", y[i]);
    printf("\n");
}
printf("\n\nCase 3: intermediate output, no root-finding\n");
for (j = 4; j <= 5; ++j) {
    tol = pow(10.0, (double) (-j));
    printf("\n Calculation with tol = %10.1e\n", tol);
    x = 0.0;
    y[0] = 0.5;
    y[1] = 0.5;
    y[2] = pi / 5.0;
    s.k = 4;
    s.h = (s.xend - x) / (double) (s.k + 1);
    printf("\n      X          Y(1)          Y(2)          Y(3)\n");

    /* nag_ode_ivp_adams_gen (d02cjc), see above. */
    nag_ode_ivp_adams_gen(neq, fcn, &x, y, s.xend, tol, Nag_Mixed, out,
                          NULLDFN, &comm, &fail);
    if (fail.code != NE_NOERROR) {
        printf("Error from nag_ode_ivp_adams_gen (d02cjc).\n%s\n",
              fail.message);
        exit_status = 1;
        goto END;
    }
}

```



```

}

printf("\n\nCase 4: no intermediate output, no root-finding");
printf(" ( integrate to xend)\n");
for (j = 4; j <= 5; ++j) {
    tol = pow(10.0, (double) (-j));
    printf("\n Calculation with tol = %10.1e\n", tol);
    x = 0.0;
    y[0] = 0.5;
    y[1] = 0.5;
    y[2] = pi / 5.0;
    printf("\n      X          Y(1)          Y(2)          Y(3)\n");
    printf("%8.2f", x);
    for (i = 0; i < 3; ++i)
        printf("%13.5f", y[i]);
    printf("\n");

    /* nag_ode_ivp_adams_gen (d02cjc), see above. */
    nag_ode_ivp_adams_gen(neq, fcn, &x, y, s.xend, tol, Nag_Mixed, NULLFN,
                          NULLDFN, &comm, &fail);
    if (fail.code != NE_NOERROR) {
        printf("Error from nag_ode_ivp_adams_gen (d02cjc).\n%s\n",
              fail.message);
        exit_status = 1;
        goto END;
    }

    printf("%8.2f", x);
    for (i = 0; i < 3; ++i)
        printf("%13.5f", y[i]);
    printf("\n");
}
END:
return exit_status;
}

static void NAG_CALL out(Integer neq, double *xsol, const double y[],
                        Nag_User *comm)
{
    Integer i;
    struct user *s = (struct user *) comm->p;

    printf("%8.2f", *xsol);
    for (i = 0; i < 3; ++i) {
        printf("%13.5f", y[i]);
    }
    printf("\n");
    *xsol = s->xend - (double) s->k * s->h;
    s->k--;
}

static void NAG_CALL fcn(Integer neq, double x, const double y[], double f[],
                        Nag_User *comm)
{
    double pwr;
    struct user *s = (struct user *) comm->p;

    if (s->use_comm[0]) {
        printf("(User-supplied callback fcn, first invocation.)\n");
        s->use_comm[0] = 0;
    }

    f[0] = tan(y[2]);
    f[1] = -0.032 * tan(y[2]) / y[1] - 0.02 * y[1] / cos(y[2]);

    pwr = y[1];
    f[2] = -0.032 / (pwr * pwr);
}

static double NAG_CALL g(Integer neq, double x, const double y[],
                        Nag_User *comm)

```

```

{
    struct user *s = (struct user *) comm->p;

    if (s->use_comm[1]) {
        printf("(User-supplied callback g, first invocation.)\n");
        s->use_comm[1] = 0;
    }

    return y[0];
}

```

## 10.2 Program Data

None.

## 10.3 Program Results

nag\_ode\_ivp\_adams\_gen (d02cjc) Example Program Results

Case 1: intermediate output, root-finding

Calculation with tol = 1.0e-04

X	Y(1)	Y(2)	Y(3)
0.00	0.50000	0.50000	0.62832
(User-supplied callback fcn, first invocation.)			
(User-supplied callback g, first invocation.)			
2.00	1.54931	0.40548	0.30662
4.00	1.74229	0.37433	-0.12890
6.00	1.00554	0.41731	-0.55068

Root of Y(1) = 0.0 at 7.288

Solution is -0.00000 0.47486 -0.76011

Calculation with tol = 1.0e-05

X	Y(1)	Y(2)	Y(3)
0.00	0.50000	0.50000	0.62832
2.00	1.54933	0.40548	0.30662
4.00	1.74232	0.37433	-0.12891
6.00	1.00552	0.41731	-0.55069

Root of Y(1) = 0.0 at 7.288

Solution is -0.00000 0.47486 -0.76010

Case 2: no intermediate output, root-finding

Calculation with tol = 1.0e-04

Root of Y(1) = 0.0 at 7.288

Solution is -0.00000 0.47486 -0.76011

Calculation with tol = 1.0e-05

Root of Y(1) = 0.0 at 7.288

Solution is -0.00000 0.47486 -0.76010

Case 3: intermediate output, no root-finding

Calculation with tol = 1.0e-04

X	Y(1)	Y(2)	Y(3)
0.00	0.50000	0.50000	0.62832
2.00	1.54931	0.40548	0.30662

4.00	1.74229	0.37433	-0.12890
6.00	1.00554	0.41731	-0.55068
8.00	-0.74589	0.51299	-0.85371
10.00	-3.62813	0.63325	-1.05152

Calculation with tol = 1.0e-05

X	Y(1)	Y(2)	Y(3)
0.00	0.50000	0.50000	0.62832
2.00	1.54933	0.40548	0.30662
4.00	1.74232	0.37433	-0.12891
6.00	1.00552	0.41731	-0.55069
8.00	-0.74601	0.51299	-0.85372
10.00	-3.62829	0.63326	-1.05153

Case 4: no intermediate output, no root-finding ( integrate to xend)

Calculation with tol = 1.0e-04

X	Y(1)	Y(2)	Y(3)
0.00	0.50000	0.50000	0.62832
10.00	-3.62813	0.63325	-1.05152

Calculation with tol = 1.0e-05

X	Y(1)	Y(2)	Y(3)
0.00	0.50000	0.50000	0.62832
10.00	-3.62829	0.63326	-1.05153

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